AVERAGE CASE ANALYSIS OF ALGORITHMS FOR THE MAXIMUM SUBARRAY PROBLEM

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by Mohammad Bashar

University of Canterbury

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Examining Committee:

- **Prof. Dr Tadao Takaoka, University of Canterbury (Supervisor)**
- **Associate Prof. Dr Alan Sprague, University of Alabama (External Examiner)**

This thesis is dedicated to
my Mom and Dad for their endless love and constant support.

Abstract

Maximum Subarray Problem (MSP) is to find the consecutive array portion that maximizes the sum of array elements in it. The goal is to locate the most useful and informative array segment that associates two parameters involved in data in a 2D array. It's an efficient data mining method which gives us an accurate pattern or trend of data with respect to some associated parameters. Distance Matrix Multiplication (DMM) is at the core of MSP. Also DMM and MSP have the worst-case complexity of the same order. So if we improve the algorithm for DMM that would also trigger the improvement of MSP. The complexity of Conventional DMM is $O(n^3)$. In the average case, All Pairs Shortest Path (APSP) Problem can be modified as a fast engine for DMM and can be solved in $O(n^2 \log n)$ expected time. Using this result, MSP can be solved in $O(n^2 \log^2 n)$ expected time. MSP can be extended to *K*-MSP. To incorporate DMM into *K*-MSP, DMM needs to be extended to *K*-DMM as well. In this research we show how DMM can be extended to *K*-DMM using *K*-Tuple Approach to solve *K*-MSP in $O(Kn^2 \log^2 n \log K)$ time complexity when $K \leq n/\log n$. We also present Tournament Approach which solves *K*-MSP in $O(n^2 \log^2 n + Kn^2)$ time complexity and outperforms the *K*-Tuple Approach significantly.

Acknowledgments

The last one and half year of my academic life has been a true learning period. This one page is not adequate to thank all those people who have been supporting me during this time.

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Chapter 1

Introduction

Maximum Subarray Problem (MSP) is to find the most useful and informative array portion that associates two parameters involved in data. It's an efficient data mining method which gives us an accurate pattern or trend of data with respect to some associated parameters. For instance, suppose we have the record of monthly sales of some products of a company and our task is to analyze the sales trend with respect to some age groups and some income levels. To formulate this into MSP, we span each sales record item over a two-dimensional array where each row and column correlates two parameters age and income level. The main objective of MSP is to detect a rectangular shaped array portion that maximizes the sum of sales. By doing so we are able to track which age groups and income levels contribute to the maximum sum of sales. Normally all the input array elements are non-negative. When this is the case, the straightforward solution is the whole array. To avoid this, we convert the given input array into a modified input array containing both positive and negative numbers. In the conversion process, we subtract the mean of the given input array elements from each single element of the given input array and consider the modified input array for the MSP. Analysis on this modified

input array would yield more precise evaluation of the sales trends with respect to age groups and income levels of purchasers.

The formulation of the problem in terms of the example matrix below is as follows:

Suppose the sales record of an arbitrary product is given in the following 2-dimensional input array.

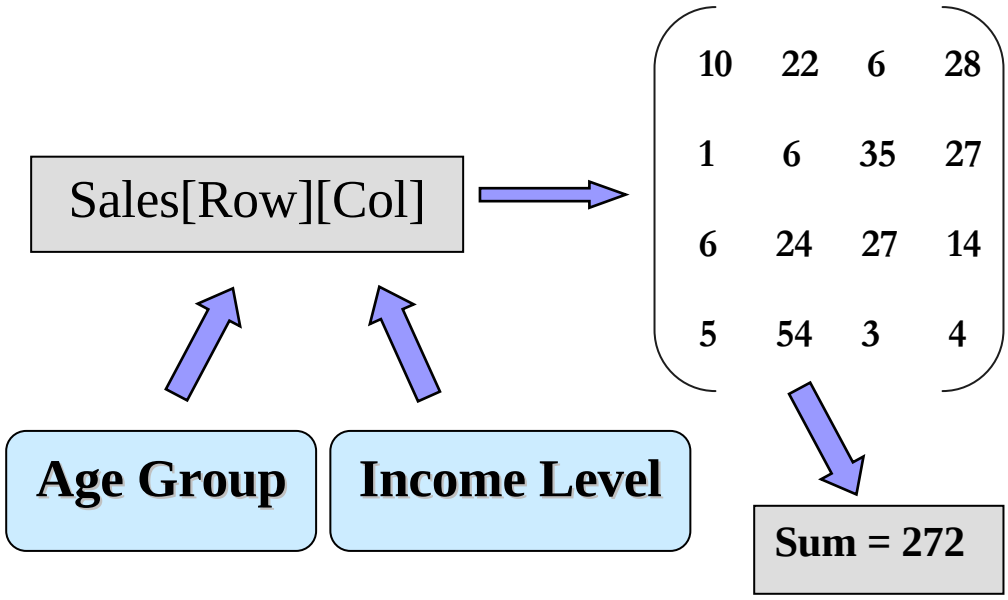


Figure 1: Given input array

If we consider the given input array with all positive numbers the trivial solution for the MSP is sum of the whole array which is 272 in this example.

As we explained earlier, to avoid this trivial solution we subtract the mean value of the array elements from each individual element of the same array. In the preceding example, the mean value of the array elements is 17 (as 272/16 = 17). In the following array we represent the modified input array after subtracting the mean value from each individual values and we find the maximum subarray on this modified input array.

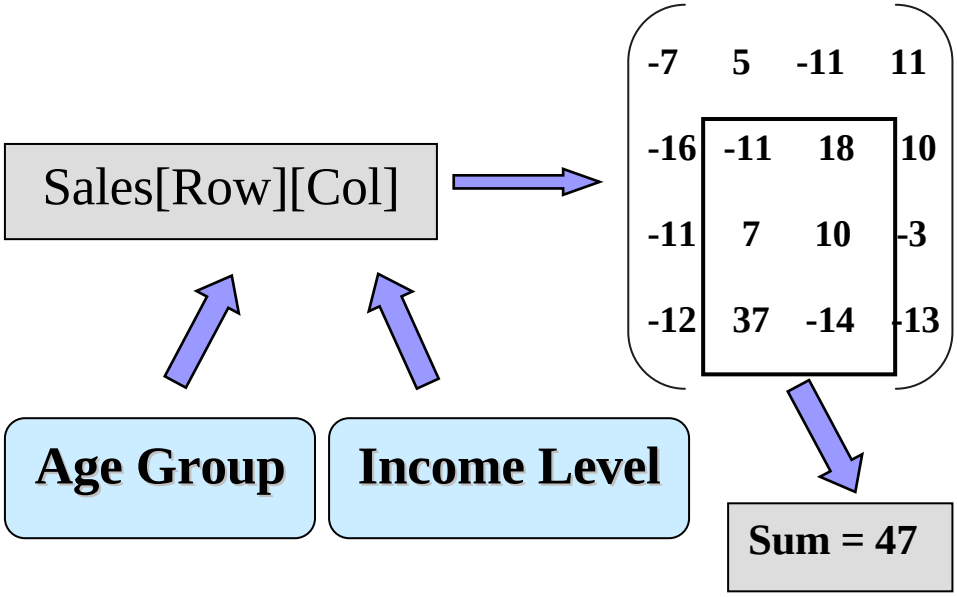


Figure 2: Converted input array

In Figure 2, we can observe the maximum subarray is defined by the rectangular array portion. The position of this subarray in the main array can be tracked as (2, 2) which corresponds to the upper left corner and (4, 3) which corresponds to the lower right corner. By tracking the maximum subarray in

this way, we can find the range of age groups and income levels that contribute to the maximum sales.

The other possible application for MSP is Graphics. The brightest portion of an image can be tracked by identifying the maximum subarray. For example, a bitmap image consists of non-negative pixel values. So we need to convert these non-negative pixel values into an accumulation of positive and negative values by subtracting the mean or average from each pixel value. By applying MSP into these modified pixel values the brightest portion in the image can be easily traced.

1.1 Maximum Subarray Problem Extension

MSP can be extended to *K*-MSP where the goal is to find *K* maximum subarrays. *K* is a positive number which is between 1 and $\frac{mn}{4}(m+1)(n+1)$ where *m* and *n* refer to the size of the given array. We explain in details in Chapter 2 with example how *K* is bounded by this number for a given array of size (*m, n*).

There are two variants of *K*-MSP. One is disjoint and the other one is overlapping. In the disjoint case, all maximum subarrays that have been detected in a given array must be separated or disjoint from each other. On the other hand, in the overlapping case, we are not enforced by such restriction.

All maximum subarrays that have been detected in a given array can be overlapped with each other.

1.2 A Real-life Significance of K- Maximum Subarray Problem

A real-life example of *K*-MSP could be a product leaflet delivery example. Let's assume we own a company and we would like to deliver information of the new products of our company on a leaflet to our potential consumers. We also would like to target our potential customers from the area of the city which is densely populated. After detecting the densely populated area of the city with the help of MSP we realized that this area is not physically accessible due to road construction work. Thus we need to identify the second most densely populated area of the city for our new products leaflet delivery. If this is also not accessible due to some reason we need to identify the third most densely populated area of the city and so on. The interesting observation here is that the first maximum subarray is available to us but practically it may not be feasible to use all the time. And thus, *K*-MSP plays its role to overcome this limitation.

1.3 Research Scope

In this research, we focus into the overlapping case of *K*-MSP. We also consider 2 dimensional MSP. We assume that the size of the array is always

power of 2. The array size is represented by *m* and *n*. If *m* and *n* are not powers of 2 we can extend our framework to the case where *m* and/or *n* are not power of 2. Also all unspecified logarithms should be considered as binary.

1.4 Research Objectives

The objectives of this research are:

1. To develop efficient algorithms for *K*-MSP.

2. To do the average case analysis of algorithms for MSP and *K*-MSP.

3. To compare and evaluate existing algorithms with new algorithms for *K*-MSP.

1.5 Research Structure

Chapter 2 of this research presents theoretical foundation of MSP as we define MSP and all its relevant concepts formally. Chapter 3 provides related works that were carried out previously by other researchers. We also rigorously analyze the existing algorithms in this chapter. Chapter 4 reveals a *K*-Tuple Approach for which we develop new algorithms for *K*-Distance Matrix Multiplication and *K*-MSP. The description and analysis of new algorithms are also presented in this chapter. Chapter 5 presents a Tournament Approach for which we further develop an algorithm for *K*-MSP. Chapter 6 scrutinizes the

experiment results and comparisons between conventional approaches and new approaches. In the end, summary and conclusion are given in Chapter 7.

Chapter 2

Theoretical Foundation

In this chapter we present a number of core concepts that are related with MSP. As we further discuss the related work of MSP in the next chapter it will become transparent how these concepts are related with MSP.

2.1 Research Assumptions

All numbers that appear in the given (*m, n*) array are random and independent of each other.

2.2 Prefix Sum

The prefix sum of an array is an array in which each element is obtained from the sum of those which precede it. For example, the prefix sum of:

(4, 3, 6, 7, 2) is (4, 4+3, 4+3+6, 4+3+6+7, 4+3+6+7+2) = (4, 7, 13, 20, 22)

For 2D cases, the prefix sum of

$$\begin{pmatrix} 1 & -5 & 9 \\ 8 & -4 & 3 \\ 4 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 5 \\ 9 & 0 & 12 \\ 13 & 7 & 17 \end{pmatrix}$$

Figure 3: Prefix sum

The prefix sums $s[1..m][1..n]$ of a 2D array $a[1..m][1..n]$ is computed by the following algorithm. Mathematically, $s[i][j]$ is the sum of $a[1..i][1..j]$.

Algorithm 1: Prefix Sum Algorithm

```
 1: for i := 0 to m
 2:    for j := 0 to n
 3:       s[i][j] = 0;
 4:       column[i][j] = 0;
 5:    end for;
 6: end for;
 7: for i := 1 to m
 8:    for j := 1 to n
 9:       column[i][j] = column[i –1][j] + a[i][j];
10:      s[i][j] = s[i][j –1] + column[i][j];
11:   end for;
12: end for;
```

The complexity of the above algorithm is straightforward. It takes $O(n^2)$ time to compute the 2D prefix sum where $m = n$.

2.3 Maximum Subarray Problem

We consider a 2D array $a[1…m, 1…n]$ as input data. The MSP is to maximize the array portion $a[k...i, l…j]$, that is, to obtain such indices (k, l) and (i, j). We assume the upper-left corner has co-ordinate (1, 1). In the example in Figure 2, (k, l) corresponds to (2, 2) and (i, j) corresponds to (4, 3).

2.3.1 Exhaustive Method

The brute force or exhaustive algorithm to calculate maximum subarray is as follows:

Algorithm 2: Exhaustive MSP Algorithm

```
1: max = – 999;
2: for  i := 1 to m
3:   for j := 1 to n
4:     for k := 1 to i
5:       for l := 1 to j
6:         currentmax = s[i][j] – s[i][l] – s[k][j] + s[k][l];
7:         if currentmax > max)
8:           max = currentmax;
9:         end if;
10:      end for;
11:    end for;
12:  end for;
13: end for;
```

We assume prefix sum is already computed by Algorithm 1 and available in array *s*. Variable *max* is initialized to –999 which resembles a negative infinite value by assuming all the numbers that appear in the given (*m*, *n*) array are between 1 and 100. Thus the range for the numbers in the modified input array becomes -100 to 100. The complexity of the above algorithm is straightforward. Because of quadruply nested for loops the complexity becomes $O(n^4)$ where $m = n$.

2.3.2. Efficient Method

Takaoka [1] described an elegant method for solving MSP. In Chapter 3 we investigate this algorithm in details. This research is primarily based on Takaoka's MSP algorithm which solves MSP by applying divide-and-conquer methodology.

2.4 K-Maximum Subarray Problem

MSP was extended to *K*-MSP by Bae and Takaoka [2]. This is an extension of the original problem. Instead of searching for only one maximum subarray we search for *K* maximum subarrays. For example, in Figure 4 we have identified 2 maximum subarrays, as *K* is 2.

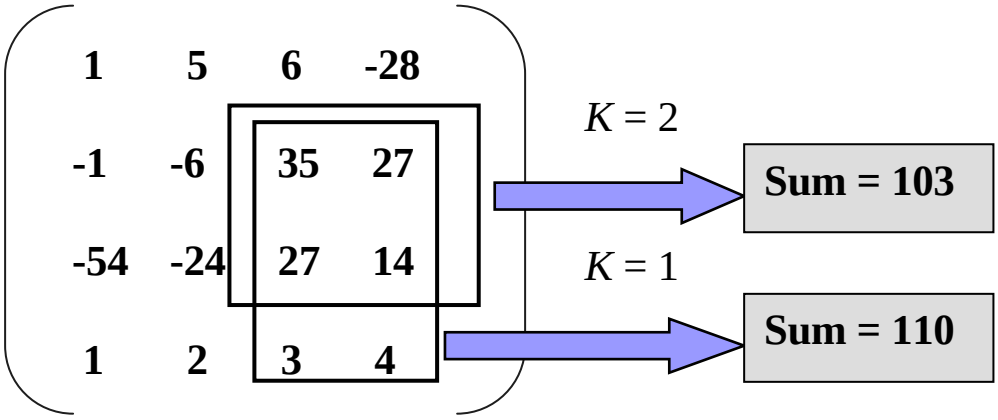


Figure 4: *K*-MSP example

2.4.1 Exhaustive Method

The brute force or exhaustive algorithm to calculate *K* maximum subarrays is as follows:

Algorithm 3: Exhaustive *K*-MSP Algorithm

```
 1:  for r := 1 to K
 2:    max[r] = – 999;
 3:    for  i := 1 to m
 4:      for j := 1 to n
 5:        for k := 1 to i
 6:          for l := 1 to j
 7:            currentmax = s[i][j] – s[i][l] – s[k][j] + s[k][l];
 8:            if (currentmax > max[r] AND currentmax's co-ordinates (i, j, k, l)
 9:                differ from all maximum subarrays co-ordinates (i, j, k, l) that
10:                 we have computed previously)
11:                 max[r] = currentmax;
12:             end if;
13:         end for;
14:       end for;
15:     end for;
16:   end for;
17: end for;
```

In the above algorithm the first condition in the '**if'** block checks for the maximum sum found for a given co-ordinate (controlled by 4 '**for'** loops). The second condition checks the co-ordinates of current sum with all previously computed sums so that the same solution is not repeated. Algorithm 3 differs from Algorithm 2 in a number of ways. The outermost '**for'** loop runs from 1

to K for K-MSP. Also *max* is a list to hold K-maximum sums. The complexity of the above algorithm is $O(Kn^4)$ where $m = n$.

2.4.2 Efficient Method

Takaoka and Bae [2] modified Takaoka's [1] MSP algorithm to deal with K-MSP. We also modify Takaoka's [1] MSP algorithm in Chapter 4 based on K-Tuple Approach. In Chapter 5, we modify Takaoka's [1] MSP algorithm once again based on Tournament Approach.

2.5 Distance Matrix Multiplication

By translating prefix sums into distances, we can solve MSP by Distance Matrix Multiplication (DMM) and we will show this in the next chapter. Now we describe DMM.

The purpose of DMM is to compute the distance product $C = AB$ for two n-dimensional matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$ whose elements are real numbers.

$$c_{i,j} = min_{k=1}^{n} \{ a_{i,k} + b_{k,j} \}$$

The meaning of $c_{i,j}$ is the shortest distance from vertex i in the first layer to vertex j in the third layer in an Acyclic Directed Graph (DAG) consisting of three layers of vertices. These vertices are labeled 1, ...,n in each layer, and the distance from i in the first layer to j in the second layer is $a_{i,j}$ and that from i in

the second layer to j in the third layer is $b_{i,j}$. The above *min* operation can be replaced by *max* operation and thus we can define a similar product, where we have longest distances in the 3-layered graph that we have described above. This graph is depicted in Figure 6.

Suppose we have the following two matrices A and B.

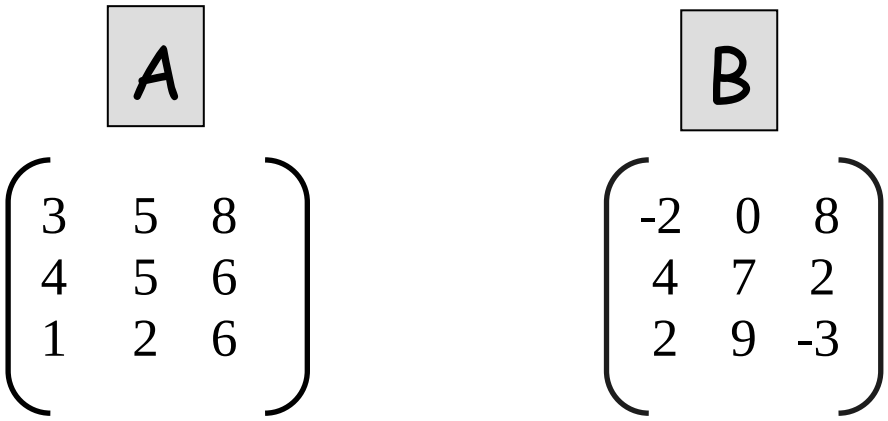


Figure 5: Two (3, 3) matrices

Then the 3-layered DAG would look like as follows:

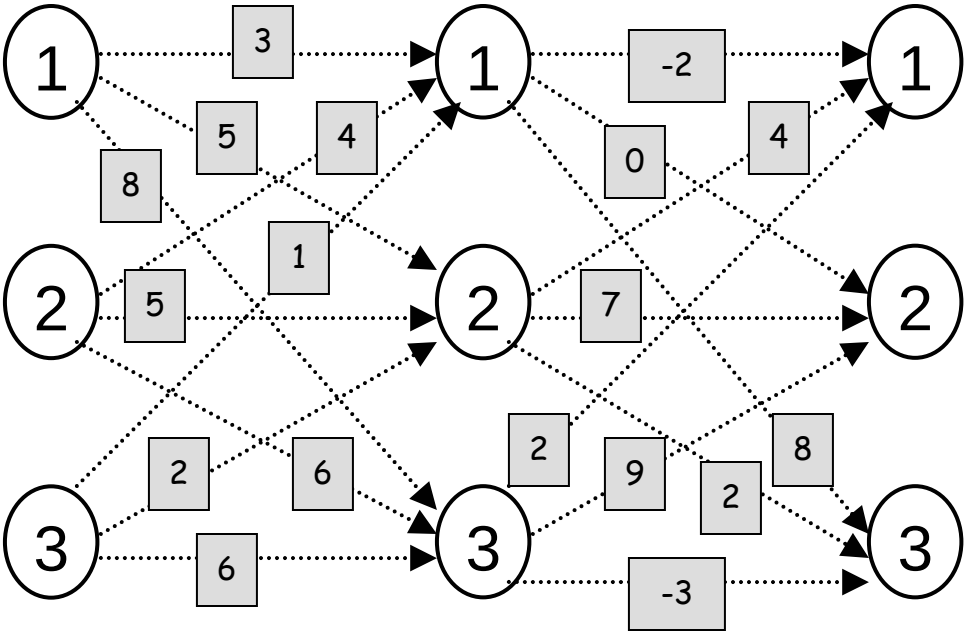


Figure 6: 3-layered DAG of two (3, 3) matrices

Then the resulting matrix *C* would look like as follows:

$$C = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 3 \\ -1 & 1 & 3 \end{pmatrix}$$

Figure 7: Resulting matrix *C*

2.5.1 Conventional DMM

The algorithm for DMM which uses the exhaustive or brute-force method is termed as Conventional DMM in this research. In the following we give the *min* version of DMM algorithm where $m = n$.

Algorithm 4: Conventional DMM Algorithm

```
 1:  for i := 1 to n
 2:    for j := 1 to n
 3:      C[i][j] = 999; //infinite value
 4:      for k :=1 to n
 5:         if(A[i][k] + B[k][j] < C[i][j] )
 6:            C[i][j] = A[i][k] + B[k][j];
 7:         end if;
 8:      end for;
 9:    end for;
10: end for;
```

The complexity of the above algorithm is $O(n^3)$ due to triple nested for loops. So this is obviously a naïve approach. For a big data set this approach would not be efficient.

2.5.2 Fast DMM

Takaoka [1] proposed All Pairs Shortest Path (APSP) Problem to be used as a fast engine for DMM. Subsequently Takaoka modified the Moffat-Takaoka's algorithm [3] for the all pairs shortest path problem (also known as MT

algorithm) and named this new algorithm as Fast DMM. This algorithm is scrutinized in Chapter 3 which takes $O(n^2 \log n)$ time.

2.6 K-Distance Matrix Multiplication

DMM can be extended to *K*-DMM as follows:

$$c_{i,j} = min\ [K]\ \{\ a_{i,k} + b_{k,j} \mid k=1..n\ \}$$

where min[*K*]S is the set of *K* minima of $S = \{\ a_{i,k} + b_{k,j} | k=1..n\ \}$. c_{ij} is now a set of *K* numbers. The intuitive meaning of *K*-DMM of MIN-version is that c_{ij} is the *K*-shortest path distances from *i* to *j* in the same graph as described before. Let $c_{ij}[k]$ be the *k*-th of c_{ij} and $C[k] = [c_{ij}[k]]$. In the above example *k* runs from 1 to 3 (as $n = 3$). So we have 3 different output matrices for C and these are shown in Figure 8, 9 and 10.

$$C[1] = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 3 \\ -1 & 1 & 3 \end{pmatrix}$$

Figure 8: Resulting matrix containing the first shortest distances

$$C[2] = \begin{pmatrix} 9 & 12 & 7 \\ 8 & 12 & 7 \\ 6 & 9 & 4 \end{pmatrix}$$

Figure 9: Resulting matrix containing the second shortest distances

$$C[3] = \begin{pmatrix} 10 & 17 & 11 \\ 9 & 15 & 12 \\ 8 & 15 & 9 \end{pmatrix}$$

Figure 10: Resulting matrix containing the third shortest distances

2.6.1 Conventional K-DMM

The exhaustive or brute-force method to find the *K*-DMM is termed as Conventional *K*-DMM algorithm in this research. In the following we give this algorithm where $m = n$.

Algorithm 5: Exhaustive *K*-DMM Algorithm

1: **for** i:= 1 to n
2: **for** j := 1 to n
3: **select** K minima of $\{a_{i1} + b_{1j},\ldots\ldots,a_{in} + b_{nj}\}$
4: **end**;
5: **end;**

We assume binary heap is used for the priority queue and insertions of new elements into the heap are done in bottom-up manner. Then line 1 and line 2 would contribute $O(n^2)$ complexity because of doubly nested for loops. On line 3, before we insert and select K items from the heap we must initialize the heap. This would cost us $O(n)$ time for n items. Then selection of K items from n items in the heap would cost us $O(K \log n)$ time. So the complexity of the above algorithm becomes $O(n^2 (n + K \log n))$ which is equivalent to $O(n^3 + n^2 K \log n)$.

2.6.2 Fast K-DMM

Takaoka's Fast DMM can be extended to find K-DMM in an efficient manner. In Fast DMM, APSP Problem is used as a fast engine where we establish the shortest path from every source to every destination. When we extend this to K-DMM, the idea is to establish K-shortest paths from every source to every destination. We modify Takaoka's Fast DMM by extending it to Fast K-DMM and we present this new algorithm in Chapter 4.

2.7 Lemmas

In this section we present a number of lemmas with their corresponding proofs which will be used through out the research.

2.7.1 Lemma 1

Suppose that in a sequence of independent trials the probability of success at each trial is p. Then the expected number of trials until the first success is $1/p$.

Proof: From the properties of the geometric distribution in [4].

2.7.2 Lemma 2

Suppose that in a sequence of independent trials the probability of each of n mutually exclusive events is $1/n$. Then the expected number of trials until all n events have occurred at least once is approximately $n \ln(n)$, where ln is natural logarithm.

Proof: From Lemma 1 and the results given in [5].

2.7.3 Lemma 3

$$\frac{1}{1-x} \leq 1 + 2x \text{ for } 0 \leq x \leq ½$$

Proof: The given condition is:

$$0 \leq x \leq ½$$

From the above given condition we can write the following:

$\Rightarrow 1 - 2x \geq 0$ or $x \geq 0$

$\Rightarrow x(1 - 2x) \geq 0$

$\Rightarrow x - 2x^2 \geq 0$

$\Rightarrow 1 + 2x - x - 2x^2 - 1 \geq 0$

$\Rightarrow (1 + 2x)(1 - x) \geq 1$ *(proved)*

2.7.4 Lemma 4

$$0 + 1 + 2 + \ldots\ldots + K - 1 = \frac{K(K-1)}{2}$$

2.7.5 Lemma 5

If an array size is (m, n) then the number of maximum subarrays is bounded by $O(m^2n^2)$ or $O(n^4)$ where $m = n$.

Proof: For an arbitrary array of size (m, n) we can establish the following inequality.

$$K \leq \sum_{i=1}^{m}\sum_{j=1}^{n} ij$$

We further simplify this inequality.

$$K \leq \sum_{i=1}^{m}\sum_{j=1}^{n} ij = \sum_{i=1}^{m} i \sum_{j=1}^{n} j = \frac{m}{2}(m+1)\,\frac{n}{2}(n+1) = \frac{mn}{4}(m+1)\,(n+1) \leq O(m^2n^2)$$

Now we consider the following example where $m = n = 2$.

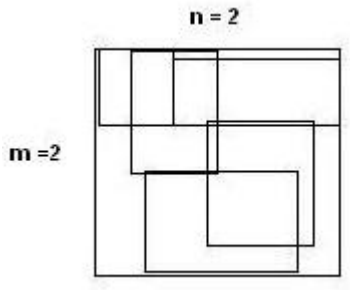


Figure 11: (m, n) array

$$K = \frac{mn}{4}(m+1)(n+1) = \frac{2x2}{4}(2+1)(2+1) = 9 \leq O(m^2n^2 = 16)$$

Using the above formula we can find the possible number of K subarrays for a given array of size (m, n).

2.7.6 Lemma 6

Part A: Now we would like to establish a formula by which we can find the required array size of (m, n) for a given K.

Using Lemma 5, we can establish the following inequality.

$$\frac{mn}{4}(m+1)(n+1) \geq K$$

If we consider the case where $m = n$ then the above inequality can be rewritten as follows:

$$\Rightarrow \frac{m^2}{4}(m+1)^2 \geq K$$

$$\Rightarrow m^2(m+1)^2 \geq 4K$$

$$\Rightarrow m(m+1) \geq 2\sqrt{K}$$

$$\Rightarrow m^2 + m - 2\sqrt{K} \geq 0$$

$$\Rightarrow m = \frac{-1 \pm \sqrt{1+8\sqrt{K}}}{2}$$

For example, if $K = 16$ we would like to find the required array size of (m, n) so that we can return 16 subarrays. We plug in $K = 16$ into the above equation and round up to the nearest integer. Since m can't be negative we only take the positive value and we get 2.37. Then we take the ceiling of this value and get $m = 3$. That is we consider $m = \left\lceil \frac{-1+\sqrt{1+8\sqrt{K}}}{2} \right\rceil$. To have K subarrays available we must have an array at least of size (3, 3). When an array size is (3,

3) there are in fact exactly 36 subarrays (using Lemma 5) available. But we only select 16 subarrays out of 36 subarrays when $K = 16$.

Part B: Let b is the exact solution for the quadratic equation. That is $b = \frac{-1+\sqrt{1+8\sqrt{K}}}{2}$. Let a = ceiling (b), which means $a < b + 1$. We can further rewrite this as $a - 1 < b$. Now using Lemma 5, we can establish the following inequality by considering the case where $m = n$ and setting $m = a - 1$ and $m = b$.

$$\frac{(a-1)^2}{4}(a-1+1)^2 < \frac{b^2}{4}(b+1)^2 < K$$

$$\Rightarrow \frac{(a-1)^2}{4}(a-1+1)^2 < K$$

$$\Rightarrow \frac{a^2}{4}(a-1)^2 < K$$

$$\Rightarrow \frac{a^2}{4}(a^2-2a+1) < K$$

$$\Rightarrow \frac{a^4-2a^3+a^2}{4} < K$$

$$\Rightarrow \frac{a^4 - 2a^3 + a^2}{4} + a^3 < K + a^3 \text{ (adding } a^3 \text{ on the both side)}$$

$$\Rightarrow \frac{a^4 + 2a^3 + a^2}{4} < K + a^3$$

$$\Rightarrow \frac{a^2(a^2 + 2a \ \ +1)}{4} < K + a^3$$

$$\Rightarrow \frac{a^2(a \ \ +1)^2}{4} < K + a^3$$

In the above we have established the fact that when we consider array size of (*a, a)* instead of exact size of (*b, b)* there are at most $K + a^3$ subarrays available.

2.7.7 Lemma 7

Endpoint independence holds for the DMM with prefix sums for a wide variety of probability distribution on the original data array.

Proof: The basic randomness assumption in this thesis is the endpoint independence which is mainly used for random graphs. In MSP randomness is defined on array variables. So here we show how the endpoint independence can be derived from the randomness assumption of the given array.

Let us take a 2 dimensional array given by $a[1][1],\ldots,a[n][n]$. Let us assume $a[i][j]$ are independent random variables with $prob\{a[i][j] > 0\} = ½$. Then we have

$prob\{a[1][j] + \ldots\ldots + a[i][j] > 0\} = ½$ ----------(I)

Also let $b[j] = a[1][j] + \ldots\ldots + a[i][j]$ ----------(II)

Let prefix sum $s[i][j] = s[i][j\text{-}1] + b[j]$

Now we ignore i and thus we can write

$s[j] = b[1] + b[2] + \ldots\ldots + b[j]$ ----------(III)

Now from (I) & (II) we have

$prob\{b[j] > 0\} = ½$ ----------(IV)

Now we consider another variable k and using (III) we can write

$s[k] = b[1] + b[2] + \ldots\ldots + b[k]$ ----------(V)

For $k < j$, from (III) & (V) we have

$s[j] - s[k] = b[k + 1] + \ldots\ldots + b[j]$

As in (IV) we have shown $b[j]$ are independent random variables with $prob\{b[j] > 0\} = ½$ then we can write

$prob\{b[k + 1] + \ldots\ldots + b[j] > 0\} = ½$ and thus

$prob\{s[k] < s[j]\} = ½$

Hence we have any permutation of $s[1], \ldots\ldots, s[n]$ with equal probability of $\frac{1}{n!}$, if we sort them in increasing or decreasing order.

Chapter 3

Related Work

In this chapter we review and discuss previous works that were carried out by other researchers on MSP.

3.1 Main Algorithm of MSP based on DMM

MSP was first proposed by Bentley [6]. Then it was improved by Tamaki and Tokuyama [8]. Bentley achieved cubic time for this algorithm. Tamaki and Tokuyama further achieved sub-cubic time for a nearly square array. These algorithms are highly recursive and complicated. Takaoka [1] further simplified Tamaki and Tokuyama's algorithm by divide-and-conquer methodology and achieved sub-cubic time for any rectangular array.

Takaoka's [1] algorithm is as follows:

Algorithm 6: Efficient MSP Algorithm

1: If the array becomes one element, return its value.
2: Otherwise, if $m > n$, rotate the array 90 degrees.
3: // Thus we assume $m \leq n$.
4: Let A_{left} be the solution for the left half.
5: Let A_{right} be the solution for the right half.
6: Let A_{column} be the solution for the column-centered problem
7: Let the solution be the maximum of those three.

The above algorithm is based on prefix sum approach. The DMM of both *min* and *max* versions are used here. Prefix sum $s[i, j]$ for array portions of $a[1..i, 1..j]$ for all i, j with boundary condition $s[i, 0] = s[0, j] = 0$ is computed and is used throughout recursion. Takaoka divided the array into two parts by the central vertical line and defined the three conditional solutions for the problem. The first is the maximum subarray which can be found in the left half, denoted as $A_{left.}$ The second is to be found on the right half, denoted as $A_{right.}$ The third is to cross the vertical center line, denoted by A_{column}. The column-centered problem can be solved in the following way.

$$A_{column} = \max_{\substack{1 \le k \le i-1 \\ 0 \le l \le n/2-1 \\ 1 \le i \le m \\ n/2+1 \le j \le n}} \{s[i, j] - s[i, l] - s[k, j] + s[k, l]\}$$

In the above we first fix i and k, and maximize the above by changing l and j. Then the above problem is equivalent to maximizing the following for $i = 1,\ldots, m$ and $k = 1,\ldots, i-1$.

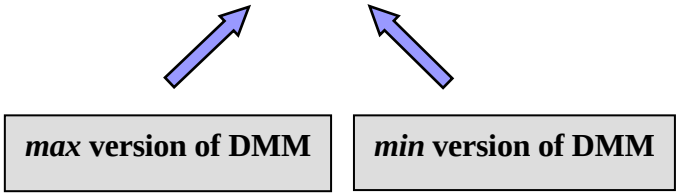
$$A_{column}[i, k] = \max_{\substack{0 \le l \le n/2-1 \\ n/2+1 \le j \le n}} \{ -s[i, l] + s[k, l] + s[i, j] - s[k, j]\}$$

Let $s^*[i, j] = -s[j, i]$. Then the above problem can further be converted into

$$A_{column}[i, k] = -\min_{0 \le l \le n/2-1} \{s[i, l] + s^*[l, k]\} + \max_{n/2+1 \le j \le n} \{s[i, j] + s^*[j, k]\} \ldots (1)$$

The first part in the above is DMM of the *min* version and the second part is of the *max* version. Let S_1 and S_2 be matrices whose (i, j) elements are $s[i, j - 1]$ and $s[i, j + n/2]$ for $i = 1..m$; $j = 1..n/2$. For an arbitrary matrix T, let T^* be that obtained by negating and transposing T. Then the above can be computed by the *min* version and taking the lower triangle, multiplying S_2 and S_2^* by the *max* version and taking the lower triangle, and finally subtracting the former from the latter and taking the maximum from the triangle. This can be expressed as

$$S_2 S_2^* - S_1 S_1^* \ldots\ldots\ldots\ldots\ldots (2)$$



3.2 The Relation between MSP and DMM

In the previous section we have observed how MSP is further converted into DMM. So DMM is at the core of MSP. Also DMM and MSP have the worst-case complexity of the same order. So if we improve the algorithm of DMM

that also triggers the improvement of MSP. That is why an efficient fast DMM algorithm can play a crucial role in the context of MSP.

3.3 APSP Problem Algorithms Lead to Fast DMM

The best known algorithm for DMM is $O(n^3 (\log \log n / \log n))$ is by Takaoka [9]. Takaoka [1] further proposed APSP Problem to be used as a fast engine for DMM. Subsequently Takaoka [1] modified the Moffat-Takaoka's algorithm [3] for APSP Problem and achieved $O(n^2 \log n)$ time complexity for DMM. Before we describe Fast DMM in detail we take a detour here to describe all APSP Problem algorithms that were involved into the development of MT algorithm and thus further contribute to the development of Fast DMM.

3.3.1 Dantzig's APSP Problem Algorithm

Dantzig [10] developed an algorithm for Single Source Shortest Path (SSSP) Problem, by which an all pairs solution is found by first sorting all of the edge lists of the graph into ascending cost order, and then solving the n single source problems. In the following, S is the solution set of vertices to which shortest distances have been established by the algorithm. Each c in S has its candidate edge (c, t). The algorithm is as follows:

Algorithm 7: Dantzig's APSP Problem Algorithm

```
1: for s = 1 to n do Single_Source(s);
2:   procedure Single_Source(s);
3:     S := {s}; d[s] := 0;
4:     initialize the set of candidates to {(s, t)}, where (s, t) is the
5:     shortest edge from s;
6:     Dantzig_expand_S(n) //the value of limit n changes in later
7:                         //algorithms
8:   end {Single_Source};
9:
10:   procedure Dantzig_expand_S(limit);
11:     while |S| < limit do begin
12:         let (c0, t0) be the candidate of least weight, where the weight
13:         of (c, t) is given by d[c] + c(c, t);
14:         S := S ∪ {t0};
15:         d[t0] := d[c0] + c(c0, t0);
16:         if |S| = limit then break;
17:         add to the set of candidates the shortest useful edge from t0;
18:         for each useless candidate (v, t0) do
19:             replace (v, t0) in the set of candidates by the next shortest
20:             useful edge from v
21:         end for;
22:     end while;
23:   end {Dantzig_expand_S};
24: end for; //end of s
```

3.3.2 Description of Dantzig's APSP Problem Algorithm

In APSP Problem we consider the problem of computing the shortest paths from a designate vertex *s*, called the source, to all other vertices in the given graph $G = (V, E)$ where V is the set of vertices, E is the set of edges, G is a complete directed graph of n vertices and we repeat this process for all

possible source vertices $s \in V$. APSP Problem also requires the tabulation of the function L, where for two vertices $i, j \in V$, $L(i, j)$ is the cost of a shortest path from i to j.

In Dantzig's algorithm, at first s is assigned a shortest path cost of zero and made the only member of a set S of labeled vertices for which the shortest path costs are known. Then, under the constraint that members c of S are "closer" to s than nonmembers u, that is, $L(s, c) \leq L(s, u)$, the set S is expanded until all vertices have been labeled. The expansion process of the solution set is shown in Figure 12.

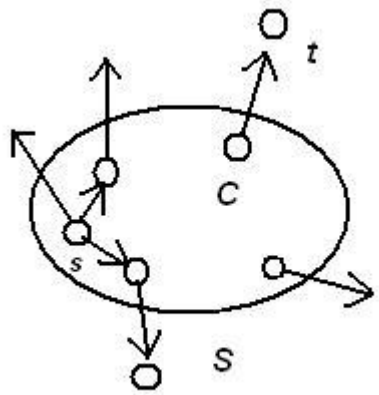


Figure 12: Solution set expansion process

The above algorithm maintains a candidate edge (c, t) for each vertex c in S. Also information is maintained about paths from vertices already in S to

vertices still outside S to make the expansion of S computationally easy. If a candidate's endpoint t is outside the current S the candidate is considered as a useful candidate and useless otherwise. Dantzig's algorithm requires all candidates to be useful. To meet this requirement the candidate for a vertex c is selected by scanning the sorted list of edges out of c in increasing cost order until a useful edge (c, t) is found. When a useful edge (c, t) is found we are guaranteed that t is the closest (by a single edge) vertex to c and $c \notin S$. Vector d maintains shortest path costs for vertices that are already labeled, that is $c \in S$, $d[c] = L(s, c)$. The edge cost from c to t is given by $c(c, t)$ and the path cost via vertex c to candidate t is given by the candidate weight $d[c] + c(c, t)$.

Vertex t might also be the candidate of other already labeled vertices $v \in S$ as in Figure 13. Each time it is a candidate there will be some weight $d[v] + c(v, t)$ associated with its candidacy. There will be in total $|S|$ candidates, with endpoints scattered amongst the $n - |S|$ unlabelled vertices $\notin S$.

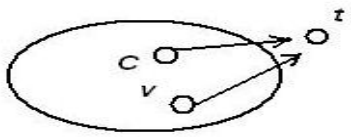


Figure 13: Two vertices pointing to same candidate

At each stage of the algorithm the endpoint of the least weight candidate is added to *S*. If c_0 is a vertex such that the candidate weight $d[c_0] + c(c_0, t_0)$ is minimum over all labeled vertices *c*, then t_0 can be included in *S* and given a shortest path cost $d[t_0]$ of $d[c_0] + c(c_0, t_0)$. Then an onward candidate for t_0 is added to the set of candidates, the candidates for vertices that have become useless are revised (including that of c_0), and the process repeats and stops when $|S| = n$. Figure 14 shows an intermediate stage of the expansion of solution set *S*.

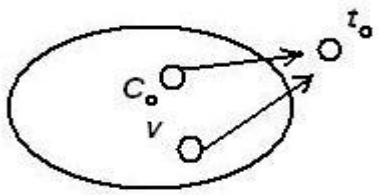


Figure 14: Intermediate stage of solution set expansion process

Initially the source *s* is made the only member of *S*, $d[s]$ is set to zero, and the candidate for *s* is the shortest edge out of *s*.

3.3.3 Analysis of Dantzig's APSP Problem Algorithm

When solution set size is *j*, that is $|S| = j$, we require $O(j)$ effort to search in an array of candidates for the candidate with the minimum cost. After we label the minimum cost candidate we require another $O(j)$ effort to check the other

candidates to decide whether or not they remain useful. Thus total effort is $O(j^2)$. When $j = n$, this is $O(n^2)$.

The other component that contributes to the running time is the effort spent scanning edge lists looking for useful edges. As each edge of the graph will be examined no more than once, the effort required for this is $O(n^2)$. Thus the total time for a single source problem becomes $O(n^2)$. And the total time for the n single source problems becomes $O(n^3)$. The time for sorting, $O(n^2 \log n)$ is absorbed within the main complexity.

3.3.4 Spira's APSP Problem Algorithm

Spira [11] also developed an algorithm for Single Source Shortest Path (SSSP) Problem in which an all pairs solution is found by first sorting all of the edge lists of the graph into ascending cost order, and then solving the n single source problems. The algorithm is as follows:

Algorithm 8: Spira's APSP Problem Algorithm

```
1: for s = 1 to n do Single_Source(s);
2:    procedure Single_Source(s);
3:       S := {s}; d[s] := 0;
4:       initialize the set of candidates to {(s, t)}, where (s, t) is the
5:       shortest edge from s;
6:       Spira_expand_S(n)
7:    end {Single_Source};
8:
9:    procedure Spira_expand_S(limit);
```

```
10:     while |S| < limit do begin
11:       let (c0, t0) be the candidate of least weight;
12:       if t0 is not in S then begin
13:         S := S ∪ + {t0};
14:         d[t0] := d[c0] + c(c0, t0);
15:         if |S| = limit then break;
16:         add to the set of candidates the shortest edge from t0;
17:       end if;
18:       replace (c0, t0) in the set of candidates by the next shortest
19:       edge from c0;
20:     end while;
21:   end {Spira_expand_S};
22: end for; //end of s
```

3.3.5 Description of Spira's APSP Problem Algorithm

Spira's *n* single source problems algorithm is quite similar to that of Dantzig. The main difference between these two algorithms is that Spira incorporated a weak candidacy rule and relaxed the strong candidacy rule that requires all candidates to be useful. The weak candidacy rule enforces that all candidate edges (c, t) be such that $c(c, t) \leq c(c, u)$ for all unlabeled vertices *u*. The main motivation behind this weak candidacy rule is that the expensive scanning of adjacency lists could be reduced. Note that in Dantzig, *t* itself must be outside *S*. To identify each successive minimal weight candidate in O(log *n*) time the set of candidates should be implemented as a binary tournament tree. The weakened candidacy rule implies that the minimal cost candidate will no longer necessarily be useful. We draw the minimal candidate at the root of the heap and replace until a useful candidate is found. Then we expand *S*.

3.3.6 Analysis of Spira's APSP Problem Algorithm

Spira introduced the cost of an increased number of candidates that must be examined during the labeling process by cutting the time spent scanning adjacency lists looking for useful edges. Spira makes an important probabilistic assumption, called endpoint independence that the minimal cost candidate falls on each of the vertices with equal probability. Using lemma 2, the total expected number of drawings of candidates will be $O(n \log n)$ until we draw all candidates at least once. Each drawing would cost us $O(\log n)$ time for the corresponding tree manipulation, thus the total effort to solve 1 single source problem is on average $O(n \log^2 n)$. And the total effort to solve the n single source problems becomes $O(n^2 \log^2 n)$.

3.3.7 MT's APSP Problem Algorithm

Moffat and Takaoka [3] further developed an algorithm which is based on Dantzig's and Spira's APSP Problem algorithms and famously known as MT algorithm. They identified a critical point until which Dantzig's APSP Problem algorithm is used for labeling vertices. After the critical point Spira's APSP Problem algorithm is used for the labeling on a subset of edges.

1
Algorithm 9: MT's APSP Problem Algorithm

1: **for** s = 1 to n **do** Fast_Single_Source(s);

```
 2:   procedure Fast_Single_Source(s);
 3:     S := {s}; d[s] := 0;
 4:     initialize the set of candidates to {(s, t)}, where (s, t) is the
 5:     shortest edge from s;
 6:     Dantzig_expand_S(n – n/log n);
 7:     U := V – S;
 8:     Spira_expand_S(n), using only the U-edges
 9:   end {Fast_Single_Source};
10: end for; //end of s
```

3.3.8 Description of MT's APSP Problem Algorithm

MT algorithm is a mixture of Dantzig's and Spira's algorithms where these algorithms are slightly modified. MT algorithm can be divided into two phases. The first phase corresponds to Dantzig's algorithm where binary heap is used for the priority queue. This phase is used to label vertices until the critical point is reached, that is when $|S| = n - n/\log n$. Dantzig incorporated the strong candidacy rule. The complexity of this algorithm consists of two components. One is the effort required to label the minimal weight candidate and to replace candidates that become useless. The other one is the effort required to scan for useful edges. The second phase after the critical point corresponds to Spira's algorithm. This is used for labeling the last $n/\log n$ vertices after the critical point. Spira incorporated the weak candidacy rule and used binary tournament tree to identify minimal weight candidates more efficiently. The weak candidacy rule does not require all candidates to be

useful. Moffat and Takaoka introduced the concept of set *U* and *U*-edge. *U* is defined as the set of unlabeled vertices after the critical point. Also an edge is called *U*-edge if it connects to a vertex in the set *U*. These are depicted in the following figure.

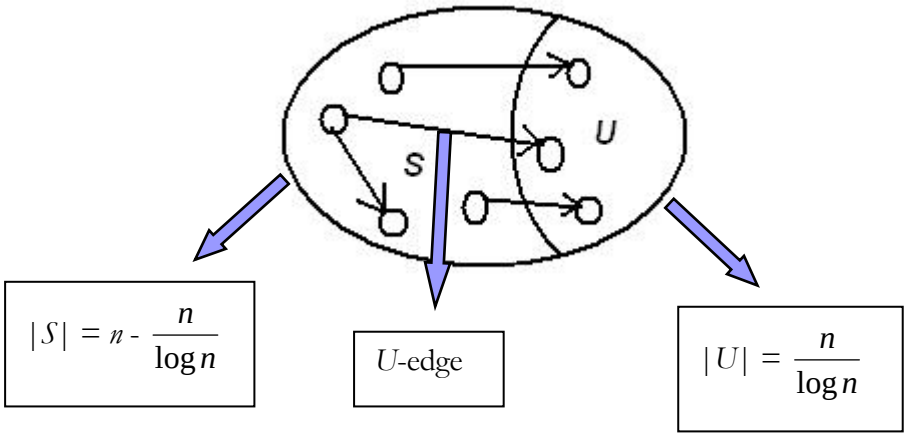


Figure 15: Visualizing set *S*, *U* and *U*-edge

To improve the efficiency of Spira's algorithm, Moffat and Takaoka made an important modification to procedure Spira_expand_*S*. Due to this modification the next shortest edge is not chosen as the new candidate, instead the next shortest *U*-edge is chosen as the new candidate.

3.3.9 Analysis of MT's APSP Problem Algorithm

Suppose at some intermediate stage of the computation there are *j* vertices that are labeled. That is $|S| = j$. So there are $n - j$ unlabeled vertices. We assume

each of the $n - j$ unlabeled vertices has the equal probability to be chosen as next candidate. Then we can expect that $1 + \frac{j-1}{n-j}$ candidates will need to be taken care of as they becomes useless in each iteration of the while loop of Dantzig_expand_*S* because by definition the root of the heap will become useless always. And each of the remaining $j - 1$ candidates will become useless with probability $p = \frac{1}{n-j}$. It be can shown that the useless candidates can be replaced in $O(\frac{j}{n-j} + \log j)$ time on average although we omit details. This expression is $O(n \log n)$ only when $j \leq n - n/\log n$ and this is how the critical point was chosen. The expansion of *S* from 1 element to $n - n/\log n$ elements will require $O(n \log n)$ expected time.

Now we focus on the scanning effort for useful edges. When $|S| = n - n/\log n$ there will be $n/\log n$ *U*-edges scattered through each edge list. Each edge list is a random permutation of endpoints. Because of this the least cost *U*-edge in each list will lie on average in the log *n*th position. Also because of the strong candidacy rule each edge list pointer will be pointing at the least cost *U*-edge in the corresponding edge list. Thus the scanning effort will require $O(n \log n)$ time on average.

Each heap operation in Spira_expand_*S* requires O(log *n*) time. Also the edge scanning effort is O(log *n*) time. One of these two efforts will be absorbed into another. Thus total effort required for a candidate replacement is O(log *n*) time. Thus each iteration of Spira_expand_*S* will require O(log *n*) time. After the critical point there are $|U| = n/\log n$ vertices that need to be labeled. As each candidate is a random member of the set *U* the expected number of while loop iterations required before all the members of set *U* gets labeled is $(n/\log n) \ln (n/\log n)$ (using Lemma 2) which is bounded by O(*n*). Thus the total expected time to label the set *U* will be O(*n* log *n*).

3.4 Modified MT Algorithm as Fast DMM

Takaoka [1] modified MT algorithm and used APSP Problem as a fast engine for DMM. In this section we describe this algorithm in details.

Algorithm 10: Fast DMM Algorithm

```
1: Sort n rows of B and let the sorted list of indices be list [1], ..., list[n];
2: Let V = {1……..n};
3: for i := 1 to n do begin
4:    for k := 1 to n do begin
5:        cand[k] := first of list[k];
6:       d[k] := a[i, k] + b[k, cand[k]];
7:    end for; //end of k
8:    Organize set V into a priority queue with keys d[1],……,d[n];
9:    Let the solution set S be empty;
10:   /* Phase 1: Before the critical point */
11:   while |S| ≤ n – n/log n do begin
12:       Find v with the minimum key from the queue;
```

```
13:       Put cand[v] into S;
14:       c[i, cand[v]] := d[v];
15:       Let W = {w | cand[w] = cand[v]};
16:       for w in W do
17:          while cand[w] is in S do cand[w] := next of list[w];
18:       end for; //end of w
19:       Reorganize the queue for W with the new keys d[w] = a[i, w] + b[w,
20:       cand[w]];
21:    end while;
22:    U := S;
23:    /* Phase 2: After the critical point */
24:   while |S| < n do begin
25:       Find v with the minimum key from the queue;
26:       if cand[v] is not in S then begin
27:          Put cand[v] into S;
28:          c[v, cand[v]] := d[v];
29:          Let W = {w | cand[w] = cand[v]};
30:       end else W = {v};
31:       for w in W do
32:          cand[w] := next of list[w];
33:          while cand[w] is in U do cand[w] := next of list[w];
34:       end for; //end of w
35:       Reorganize the queue for W with the new keys d[w] = a[i, w] + b[w,
36:       cand[w]];
37:    end while;
38: end for; //end of i
```

3.4.1 Description of Fast DMM Algorithm

Suppose we are given two distance matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$. The main objective of DMM is to compute the distance product C of A and B. That is $C = AB$. In this description of the algorithm, suffices are represented by brackets. At first rows of B need to be sorted in increasing order. Then we solve the n single source problems by MT algorithm by using the sorted lists of indices

list[*k*]. We solve the single source problem first and repeat it *n* times for *n* different sources. The endpoint independence is assumed on the lists list[*k*] using Lemma 7, that is, when we scan the list, any vertex can appear with equal probability. We consider the 3-layered DAG in Figure 16 in further description of this algorithm.

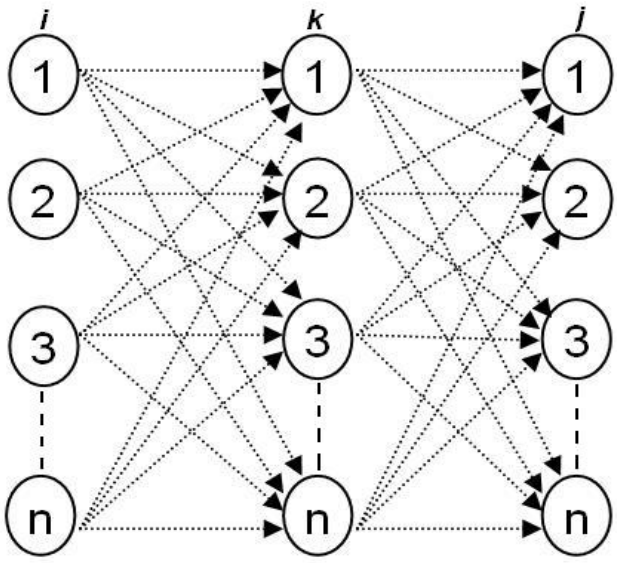


Figure 16: 3-layered DAG of two (*n*, *n*) matrices

From source *i* in the first layer, let each vertex *k* in the second layer have its candidate *cand*[*k*] in the third layer, which is the first element in *list*[*k*] initially. All the second layer vertices {*k* | *k* = 1,….,*n*} is organized into a priority queue by the keys *d*[*k*] = *a*[*i*, *k*] + *b*[*k*, cand[*k*]]. The process of deletion of *v* with the minimum key from the queue is repeated and *cand*[*v*] is

inserted into the solution set *S*. list[*v*] is scanned to get a clean candidate for *v* so that *cand*[*v*] points to a vertex which is outside the current solution set *S*. Then *v* is inserted back to the queue with the new key value. Every time the solution set is expanded by one, we scan the lists for other *w* such that *cand*[*w*] = *cand*[*v*] and construct the set *W*. For each $w \in W$ we forward the corresponding pointer to the next candidate in the list to make their candidates *cand*[*w*] clean. The key values are changed accordingly and we reorganize the queue according to new key values. This expansion process of the solution set stops at the critical point where $|S| = n - n/\log n$. We consider *U* to be the solution set at this stage and *U* remains unchanged from this point onwards. After the critical point, solution set is further expanded to *n* in the same way to label rest of the $n/\log n$ candidates which are outside *U*.

3.4.2 Analysis of Fast DMM Algorithm

Analysis of Fast DMM, not surprisingly, should be quite similar to that of MT. First we focus on heap operations. If a binary heap is used for the priority queue, and the reorganization of the heap is done for *W* in a bottom-up fashion then the expected time for reorganization can be shown to be $O(n/(n-j) + \log n)$, when $|S| = j$. This expression is bounded by $O(\log n)$ when $|S| \leq n - n/\log n$. Thus the effort requires for reorganization of the queue in phase 1 is $O(n \log n)$ in total.

Using Lemma 2, we can establish the fact that after the critical point the expected number of unsuccessful trials before we get $|S| = n$ is (n/log n) ln (n/log n). This is bounded by O(n). The expected size of W in phase 2 is O(log n) when S is expanded, and 1 when it is not expanded. The queue reorganization is done differently in phase 2 by performing increase-key separately for each w, spending O(log n) time per cand(w). From these facts, the expected time for the queue reorganization in phase 2 can be shown to be

$$O(\frac{n}{\log n} \times \log^2 n) = O(n \log n).$$

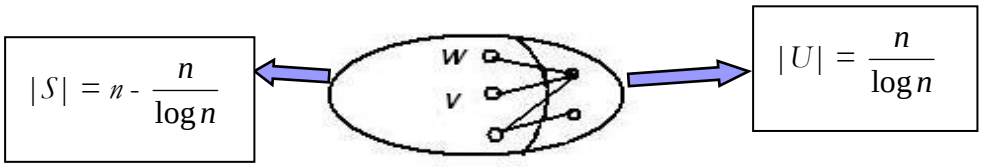


Figure 17: Intermediate stage of set W

It can be shown that the scanning efforts to get clean candidates in phase 1 and in phase 2 are both O(n log n) by using the same technique discussed in section 3.3.9 for the analysis of MT algorithm. From these observations it can be concluded that complexities before and after the critical point are balanced to be O(n log n), resulting in the total expected time of O(n log n). Thus expected time for the n single source problems becomes O(n^2 log n). The time for sorting is absorbed with in the main complexity.

3.5 Analysis of MSP (Algorithm 6) based on Fast DMM

We assume m and n are each a power of 2, and $m \leq n$. By chopping the array into squares we can consider the case where $m = n$. Let $T(n)$ be the time to analyze an array of size (n, n). Algorithm 6 splits the array first vertically and then horizontally. We multiply $(n, n/2)$ and $(n/2, n)$ matrices by 4 multiplications of size $(n/2, n/2)$ and analyze the number of comparisons. Let $M(n)$ be the time for multiplying two $(n/2, n/2)$ matrices which is equal to $O(n^2 \log n)$ as this is the expected time for the n single source problems by Fast DMM. Thus we can establish the following lemma, recurrence and theorem.

3.5.1 Lemma 8

If $M(n) = O(n^2 \log n)$ then $M(n)$ satisfies the following condition:

$$M(n) \geq (4 + 4/\log n)M(n/2)$$

Proof:

$$M(n) = n^2 \log n$$

$$= 4\left(\frac{n}{2}\right)^2 \log\left(2.\frac{n}{2}\right)$$

$$= 4\left(\frac{n}{2}\right)^2 \left(1 + \log\frac{n}{2}\right)$$

$$= 4\left(\frac{n}{2}\right)^2 \log\frac{n}{2}\left(1 + \frac{1}{\log\frac{n}{2}}\right)$$

$$= \left(4 + \frac{4}{\log \frac{n}{2}}\right) \mathrm{M}\left(\frac{n}{2}\right) > \left(4 + \frac{4}{\log n}\right) \mathrm{M}\left(\frac{n}{2}\right)$$

Thus $M(n)$ satisfies the condition.

3.5.2 Recurrence 1

Let T_1, T_2,….., T_N be the times to compute DMMs of different sizes in the Algorithm 6. Then by ignoring some overhead time between DMMs, the expected value $E[T]$ of the total time T becomes

$E[T] = E[T_1 + T_2 + \ldots\ldots + T_N]$

$= E[T_1] + E[T_2] + \ldots\ldots + E[T_N]$

$= E[E[T_1 \mid T_2, T_3, \ldots\ldots, T_N]] + E[E[T_2 \mid T_1, T_3, \ldots\ldots, T_N]] + \ldots\ldots + E[E[T_N \mid$

$T_1, T_2, \ldots\ldots, T_{N-1}]]$

From the theorem of total expectation, we have $E[E[X|Y]] = E[X]$ where $X|Y$ is the conditional random variable of X conditioned by Y. So we can write

$E[T] = E_{T_1}[T_1] + E_{T_2}[T_2] + \ldots\ldots + E_{T_N}[T_N]$

In the above, expectation operators go over the sample space of T_1, T_2,….., T_N. Thus we can establish the following recurrence where total expected time is expressed as the sum of the expected times of algorithmic components.

$T(1) = 0$, $T(n) = 4T(n/2) + 12M(n)$

$M(n) = O(n^2 \log n)$

3.5.3 Theorem 1

Suppose $M(n) = O(n^2 \log n)$ and $M(n)$ satisfies the condition $M(n) \geq (4 + 4/\log n)M(n/2)$. Then the above $T(n)$ satisfies the following:

$T(n) \leq 12(1 + \log n)M(n)$

Proof:

Basis step: Theorem 1 holds for $T(1)$ from the algorithm.

Inductive step:

$T(n) = 4T(n/2) + 12M(n)$

$$= 4 \times 12(1 + \log \frac{n}{2}) \times M(\frac{n}{2}) + 12M(n)$$

$$\leq 48 \times \frac{1+\log\frac{n}{2}}{4+\frac{4}{\log n}} \times M(n) + 12M(n)$$

$$= 48 \times \frac{1+\log n - \log 2}{4(1+\frac{1}{\log n})} \times M(n) + 12M(n)$$

$$= 12 \times \frac{1+\log n - 1}{\frac{\log n + 1}{\log n}} \times M(n) + 12M(n)$$

$$= 12 \times \frac{\log n}{\frac{\log n + 1}{\log n}} \times M(n) + 12M(n)$$

$$= 12 \times \frac{\log^2 n}{\log n + 1} \times M(n) + 12M(n)$$

$$= 12 \times \frac{\log^2 n + \log n - \log n}{\log n + 1} \times M(n) + 12M(n)$$

$$= 12 \times \frac{\log n(\log n + 1) - \log n}{\log n + 1} \times M(n) + 12M(n)$$

$$= 12 \times \left(\log n - \frac{\log n}{\log n + 1} \right) \times M(n) + 12M(n)$$

$\leq 12 \log n \, M(n) + 12M(n)$

$= 12M(n) (\log n + 1) = 12(1 + \log n)M(n)$ *(proved)*

Thus the total complexity of Algorithm 6 based on Fast DMM becomes $O(n^2 \log^2 n)$. Because an extra log n effort is required on top of the Fast DMM complexity due to the recursion in Algorithm 6.

3.6 K-MSP Algorithm

Takaoka and Bae [2] developed an algorithm for *K*-MSP incorporating sub-cubic DMM algorithm. Since this sub-cubic DMM algorithm is beyond the scope of this research we are not going to analyze this algorithm.

Chapter 4

K-Tuple Approach

DMM can be generalized as *K*-DMM as we already explained this in Chapter 2 (Sec 2.6). In this chapter we modify Takaoka's Fast DMM based on MT algorithm where APSP Problem is incorporated as a fast engine for DMM. In the following algorithm, we extend the Single Source Shortest Path Problem to the Single Source *K* Shortest Paths Problem and establish Fast *K*-DMM algorithm by incorporating *n* such problems.

In the following algorithm we have the outermost loop by *r*. Data structure *T* is used to identify edges already consumed in shortest paths for previous *r*. *T* can be implemented as a 2D array where at every location we maintain a set of second layer vertices of the 3-layered DAG through which a path has been established for a given pair of source and destination.

4.1 New Fast K-DMM Algorithm

Algorithm 11: Fast *K*-DMM Algorithm

1: Sort *n* rows of *B* and let the sorted list of indices be *list*[1], ...,*list*[*n*]
2: Let $V = \{1……..n\}$;
3: Let the data structure *T* be empty;
4: **for** *r* := 1 **to** *K* **do begin**
5: **for** *i* := 1 **to** *n* **do begin**

```
6:    for k := 1 to n do begin
7:      cand[k] := first of list [k];
8:      while k is found in the set at T[i, cand[k]] location  do begin
9:          cand[k] := next of list[k];
10:         if end of list is reached then begin
11:             cand[k] := n + 1; //(n + 1)th column of b holds
11:                                 // a positive infinite value 999
12:         end if;
13:     end while;
14:     d[k] := a[i, k] + b[k, cand[k]];
15:   end for; //end of k
16:   Organize set V into a priority queue with keys d[1],……, d[n];
17:   Let the solution set S be empty;
18:   /* Phase 1: Before the critical point */
19:   while |S| ≤ n – n/ log n do begin
20:     find v with the minimum key from the queue;
21:     Put cand[v] into S;
22:     c[r, i, cand[v]] := d[v];
23:     Put v in the set at T[i, cand[v]] location;
24:     Let W = {w | cand[w] = cand[v]};
25:     for w in W do begin
26:       while cand[w] is in S OR w is found in the set at T[i, cand[w]]
27:        location do begin
28:         cand[w] := next of list[w];
29:         if end of list is reached then begin
30:           cand[w] := n + 1; //(n + 1)th column of b holds
31:                                 // a positive infinite value 999
32:         end if;
33:       end while;
34:     end for; // end of w
35:     Reorganize the queue for W with the new keys d[w] = a[i, w] + b[w,
36:     cand[w]];
37:   end while;
38:   U := S;
39:   /* Phase 2: After the critical point */
40:   while |S| < n do begin
41:     find v with the minimum key from the queue;
42:     if cand[v] is not in S then begin
43:       Put cand[v] into S;
44:       c[r, i, cand[v]] := d[v];
45:       put v in the set at T[i, cand[v]] location;
```

```
46:          Let W = {w | cand[w] = cand[v]};
47:        end else W = {v};
48:        for w in W do begin
49:          cand[w] := next of list[w];
50:          while cand[w] is in U OR w is found in the set at T[i, cand[w]]
51:          location do begin
52:            cand[w] := next of list[w];
53:            if end of list is reached then begin
54:              cand[w] := n + 1 //(n + 1)th column of b holds
55:                                  // a positive infinite value 999
56:            end if;
57:          end while;
58:        end for // end of w
59:        Reorganize the queue for W with the new keys d[w] = a[i, w] + b[w,
60:        cand[w]];
61:      end while;
62:    end for; // end of i
63:  end for;// end of r
```

4.2 Description of New Fast K-DMM Algorithm

The new Fast *K*-DMM algorithm is primarily based on Fast DMM algorithm (Algorithm 10). Some of the descriptions of Fast *K*-DMM algorithm should be quite similar to that of Fast DMM algorithm. So we omit similar descriptions here for Fast *K*-DMM algorithm that we've already discussed for Fast DMM algorithm and only focus on the new enhancements that we have made into Fast *K*-DMM algorithm.

Let $A = [a_{i,j}]$ and $B = [b_{i,j}]$ be the two distance matrices. Let $C = AB$ be the distance product. To represent these two distance matrices we declare two 2D

arrays $a[1..n, 1..n]$ and $b[1..n, 1..n + 1]$. Note that each row of b has $n + 1$ columns. This is because we need an extra position in each row to store a positive infinite value.

In this description we consider the 3-layered DAG in Figure 16. The variable r in the outermost for loop runs from 1 to K which determines the number of shortest paths for the n single source problems. As there are at most n numbers of different paths from a source to a destination, K must be $\leq n$. Also we put 999 as a positive infinite value (as we assume all the numbers that appear in the given (m, n) array are between 1 and 100) at the $(n + 1)^{th}$ column of all the rows of array b. To explain this let's consider the following situation.

Suppose for a vertex k in the second layer we have exhausted all possible candidates in the third layer. That means all possible paths from a source to a destination through vertex k have been established and there are no more candidates available in the candidate list for vertex k. In this situation we should avoid establishing any new path through vertex k for the same source and destination. So when we reach the end of the candidate list we return a special candidate with positive infinite key value for vertex k so that this key value never appears at the root of the heap and thus we do not establish any new path through vertex k for the same source and destination. So we set (n +

1)th position as the candidate of k which in return will access the $(n + 1)^{th}$ column of the k^{th} row where we have 999 as a positive infinite value.

We also maintain a data structure T to manage second layer vertices through which paths have been already established. We call second layer vertex as *via vertex*. In the next section we explain in details how we implement data structure T.

4.2.1 Description of Data Structure T

To explain the data structure T we consider the 3-layered graph in Figure 6. For example, when we finalize the distance vector for source 1(in the first layer) and destination 2 (in the third layer) via 1 (in the second layer) we set the value as 3 + 0 = 3. To calculate K shortest paths for a pair of source and destination we must keep track of *via vertices* in the second layer through which we establish shortest paths. To keep track of *via vertices*, we manipulate data structure T as a 2D array $T[1..n, 1..n + 1]$. At each position of array T we construct one empty Binary Search Tree (BST) initially. During the finalization of a distance vector for a pair of source and destination we insert the vertex of the second layer (through which the path has been finalized) into the BST of the source and destination position of array T. Note that the second dimension of T is $n + 1$ instead of n. It's because we need one extra empty position along the second dimension of T to skip '**while'** loops on line 8, 26

and 50 in Algorithm 11 in the situation when reach end of the candidate list for a second layer vertex *k*.

In the preceding example, at *T*[1, 2] position we construct a BST and insert the value of the *via vertex* which is 1 in this example. In this way, when we calculate the next shortest distance for the same pair of *source* and *destination* we consult the BST at *T*[*source*][*destination*] position and verify whether the *via vertex* of the second layer that we have chosen already exists or not in the BST. If the *via vertex* is found in the BST that reveals a path has been already established through this *via vertex* for this pair of *source* and *destination*. So we must avoid choosing the same path for the same pair of *source* and *destination* when we calculate the next shortest path for that pair of *source* and *destination*.

By consulting the BST at *T*[*source*][*destination*] position we can take the decision whether a particular *via vertex* for a pair of *source* and *destination* is valid or not. We can visualize the array *T* as in Figure 18.

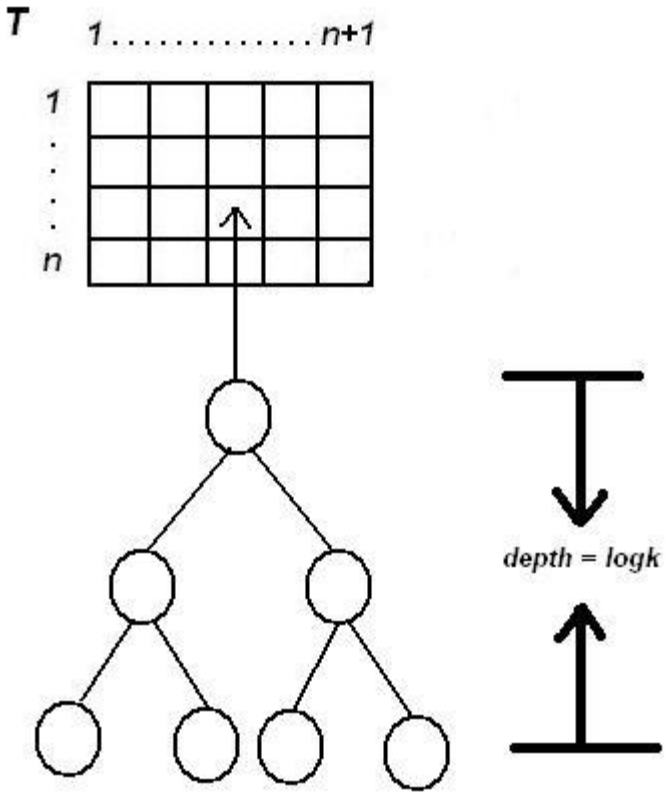


Figure 18: Visualization of array *T*

Every time we finalize a path from source vertex *i* in the first layer through the vertex *k* in the second layer to the destination vertex *j* in the third layer we put the corresponding *k* in the BST at *T*[*i*][*j*].

4.2.2 Description of Set W

We maintain set *W* to represent the concept of '*useless candidates*' of Dantzig's APSP Problem algorithm. During the expansion process of the

solution set when a new vertex in the third layer gets added to the solution set eventually it becomes useless candidate for vertices in the second layer. So all the vertices in the second layer that are pointing to this recently added vertex to the solution set must be taken care of so that they point to a clean candidate rather than a *useless* candidate. We construct set *W* to represent the vertices in the second layer that are pointing to useless candidates in the third layer.

When we update set *W* we update every member of *W* with new candidates based on two conditions. The first condition is to check whether the updated candidate for *W* already exists in the set *S*. The second condition checks for *via vertex* collision in the BST. That is to check whether *w* already exists in the BST at *T*[*i*][*cand*[*w*]] where *i* represents the current source. These two conditions are checked inclusively. That is, if any of these two conditions is true we update the candidate for current w with the next member of the *list*[*w*]. When we hit the end of the list by not being able to find any new candidate that makes both conditions false, we set $n + 1$ position of the *list*[*w*] as new candidate of *w*. As we have a positive infinite value 999 at this position this candidate will never be chosen from the heap to avoid repetition of any of the *K* shortest paths of the *n* single source problems.

4.3 Analysis of New Fast K-DMM Algorithm

The new algorithm can be divided into two parts. One is before the critical point and the other one is after the critical point. Before the critical point we employ Dantzig's APSP Problem algorithm by incorporating binary heap for the priority queue. This phase is used to label vertices to indicate they are in S until the critical point when $|S| = n - n/\log n$. After the critical point Spira's APSP Problem algorithm is used for the labeling but on a subset of edges. Spira's APSP Problem algorithm also incorporates Binary heap for the priority queue.

The endpoint independence is assumed on the lists *list*[*k*] using Lemma 7, that is, when we scan the list, any vertex can appear with equal probability.

We also assumed a binary heap is used for the priority queue through out the algorithm and the reorganization of the heap is done in a bottom-up manner.

4.3.1 Phase 1: Dantzig's Counterpart

For phase 1, let us assume $|S| = j$. So when we finalize a distance and add a vertex into the solution set the probability that some other vertex in the second layer pointing to the vertex in the third layer which has been added to the solution set is $\frac{1}{n-j}$. And we have such n vertices in the second layer. Let P

represents the expected number of affected vertices in the second layer. Then P is $\frac{n}{n-j}$. And time to detect the vertices which must be taken care of in the heap is bounded by O(log n) since $|S| \leq n - n \log n$. Thus the effort for the queue reorganization in phase 1 for the first iteration of r is:

$$\sum_{j=1}^{n-n/\log n} O(\frac{n}{n-j} + \log j) \leq O(n \log n)$$

Proof:

We can separate the above inequality into two parts.

$$(\sum_{j=1}^{n-n/\log n} O(\frac{n}{n-j}) + \sum_{j=1}^{n-n/\log n} \log j) \leq O(n \log n)$$

Let $A^1 = \sum_{j=1}^{n-n/\log n} O(\frac{n}{n-j})$ and $A^2 = \sum_{j=1}^{n-n/\log n} \log j$

Thus we need to prove:

$A^1 + A^2 \leq O(n \log n)$

Let's do part A^1 and A^2 separately.

Part A^1:

$$A^1 = \sum_{j=1}^{n-n/\log n} O(\frac{n}{n-j}) \leq \sum_{j=1}^{n-1} \frac{n}{n-j}$$ (we ignore the big 'O' notation)

$$= n \sum_{j=1}^{n-1} \frac{1}{n-j}$$

$$= n \left(\frac{1}{n-1} + \frac{1}{n-2} + \ldots\ldots\ldots + \frac{1}{n-(n-1)}\right)$$

$= n (\log n - \gamma)$ (γ is Euler's [12] Constant which is

equal to 0.577215665)

$$= n \log n - n\gamma$$

$$= O(n \log n - n\gamma)$$

Part A^2:

$$A^2 = \sum_{j=1}^{n-n/\log n} \log j \leq \sum_{j=1}^{n} \log j$$

$$= \log 1 + \log 2 + \ldots\ldots\ldots + \log n$$

$$= \log (1 \times 2 \times \ldots\ldots\ldots \times n)$$

$$= \log (n!)$$

$= \log\ n^n\ e^{-n}$ (Stirling's [13] Formula)

$= n \log n - n \log e$

$= O(n \log n - n \log e)$

L.H.S. $= A^1 + A^2$

$= O(n \log n - n\gamma) + O(n \log n - n \log e) \leq O(n \log n)$ *(proved)*

Thus the effort to reorganize the queue in phase 1 for the n single source K shortest paths problems is $\leq O(Kn^2 \log n)$.

We now focus into the scanning efforts to get clean candidates in phase 1. Let us assume $|S| = j$. The probability to get a clean candidate is $\frac{n - j - (r-1)}{n}$. Up to critical point we can establish the following inequality:

$j \leq n - n/\log n$

$\Rightarrow -j \geq -n + n/\log n$

$\Rightarrow n - j \geq n - n + n/\log n$

$\Rightarrow n - j - (r-1) \geq n/\log n - (r-1)$

$$\Rightarrow \frac{n-j-(r-1)}{n} \geq \frac{n/\log n-(r-1)}{n}$$

$$\Rightarrow \frac{n-j-(r-1)}{n} \geq \frac{1}{\log n} - \frac{r-1}{n}$$

Using Lemma 1, the expected number of trials to get one single clean candidate is:

$$\frac{n}{n-j-(r-1)} \leq \frac{1}{\frac{1}{\log n} - \frac{r-1}{n}}$$

$$= \frac{\log n}{1-\frac{\log n(r-1)}{n}} \text{ (Multiply numerator and denominator by log } n\text{)}$$

$$= \log n \left(\frac{1}{1-\frac{\log n(r-1)}{n}} \right)$$

The expected number of trials to get $n - n/\log n$ clean candidates is:

$$\sum_{j=1}^{n-n/\log n} \frac{n}{n-j-(r-1)} \leq \sum_{j=1}^{n} \frac{n}{n-j-(r-1)}$$

$$= \sum_{j=1}^{n} \log n \left(\frac{1}{1 - \frac{\log n(r-1)}{n}} \right)$$

$$= n \log n \left(\frac{1}{1 - \frac{\log n(r-1)}{n}} \right)$$

For a BST with K vertices we need to spend O(log K) time to verify a particular vertex's existence in the BST. So we multiply by log K.

The expected number of trials to get $n - n/\log n$ clean candidates for 1 single source is $\leq n \log n \log K \left(\frac{1}{1 - \frac{\log n(r-1)}{n}} \right)$

Thus the expected number of trials to get $n - n/\log n$ clean candidates for n single source is $\leq n^2 \log n \log K \left(\frac{1}{1 - \frac{\log n(r-1)}{n}} \right)$.

Using Lemma 2 and setting $x = \frac{\log n(r-1)}{n}$ we can write:

$$n^2 \log n \log K \left(\frac{1}{1 - \frac{\log n(r-1)}{n}} \right) \leq n^2 \log n \log K \left(1 + 2 \times \frac{\log n(r-1)}{n} \right)$$

where $0 \leq \frac{\log n(r-1)}{n} \leq \frac{1}{2}$

Thus the expected number of trials to get $n - n/\log n$ clean candidates for the n single source problems of 1 iteration of r is:

$$\leq T_r = n^2 \log n \log K \left(1 + 2 \times \frac{\log n(r-1)}{n} \right)$$

We take summation of $r = 1$ to K for K iterations.

$$\sum_{r=1}^{K} T_r = \sum_{r=1}^{K} n^2 \log n \log K \left(1 + 2 \times \frac{\log n(r-1)}{n} \right)$$

$$= \sum_{r=1}^{K} n^2 \log n \log K + (n \log^2 n \log K \times 2(r-1))$$

Let $A^1 = n^2 \log n \log K$ and $A^2 = n \log^2 n \log K \times 2(r-1)$

Thus we can write the following:

$$= \sum_{r=1}^{K} A^1 + A^2$$

$$= \sum_{r=1}^{K} A^1 + \sum_{r=1}^{K} A^2$$

Let's do part A^1 and A^2 separately.

Part A^1:

$$\sum_{r=1}^{K} A^1 = \sum_{r=1}^{K} n^2 \log n \log K$$

$$= Kn^2 \log n \log K$$

Part A^2:

$$\sum_{r=1}^{K} A^2 = \sum_{r=1}^{K} n \log^2 n \log K \times 2(r-1)$$

$$= n \log^2 n \log K \times 2 \times \frac{K(K-1)}{2} \text{ (using Lemma 4)}$$

$$= n \log^2 n \log K \times (K^2 - K)$$

$$= K^2 n \log^2 n \log K - Kn \log^2 n \log K$$

$$\leq K^2 n \log^2 n \log K$$

We put part A^1 and A^2 together and get

$$\sum_{r=1}^{K} T_r = O(Kn^2 \log n \log K + K^2 n \log^2 n \log K)$$

$$= O(Kn^2 \log n \log K) \text{ when } K \leq n/\log n$$

Finally we can claim the expected number of trials to get $n - n/\log n$ clean candidates for the n single source problems of K iteration of r is $\leq O(Kn^2 \log n \log K)$ when $K \leq n/\log n$.

4.3.2 Phase 2: Spira's Counterpart

By the coupon collector's problem in [4] we need to collect $O(m \log m)$ coupons in average before we get m different types of coupons. After critical point in phase 2 we need to collect $n/\log n$ vertices in the solution set. By setting $m = n/\log n$ we need to collect $(n/\log n) \log (n/\log n) \leq n$ vertices before we get $|S| = n$. Each iteration in phase 2 takes $O(\log n)$ time, resulting in $O(n \log n)$ time for 1 single source problem and $O(n^2 \log n)$ time for the n single source problems. Thus the effort for the queue reorganization in phase 2 for the n single source K shortest paths problems is $\leq O(Kn^2 \log n)$

Now we focus into the scanning efforts to get unlabelled candidates in phase 2. Let us assume $|S| = n - n/\log n$. Thus the probability to get an unlabelled candidate is $\dfrac{\frac{n}{\log n} - (r-1)}{n}$.

Now we establish the following inequality.

$\log n - 1 \leq \log n$

$\Rightarrow \log n \leq \log n + 1$

$\Rightarrow \frac{1}{\log n} \geq \frac{1}{\log n + 1}$

$\Rightarrow \frac{n}{\log n} \geq \frac{n}{\log n + 1}$ (Multiply both sides by n)

$\Rightarrow \frac{n}{\log n} - (r-1) \geq \frac{n}{\log n + 1} - (r-1)$ (Subtract r –1 from both sides)

$\Rightarrow \frac{\frac{n}{\log n} - (r-1)}{n} \geq \frac{\frac{n}{\log n + 1} - (r-1)}{n}$ (Divide both sides by n)

$\Rightarrow \frac{\frac{n}{\log n} - (r-1)}{n} \geq \frac{1}{\log n + 1} - \frac{r-1}{n}$

Using Lemma 1, the expected number of trials to get one unlabelled candidate is:

$$\frac{1}{\frac{\frac{n}{\log n}-(r-1)}{n}} \le \frac{1}{\frac{1}{\log n+1}-\frac{r-1}{n}}$$

$$\Rightarrow \frac{n}{\frac{n}{\log n}-(r-1)} \le \frac{1}{\frac{1}{\log n+1}-\frac{r-1}{n}}$$

$$= \frac{\log n+1}{(\log n+1)\left(\frac{1}{\log n+1}-\frac{r-1}{n}\right)}$$ (Multiply Numerator and Denominator by $\log n+1$)

$$= \frac{\log n+1}{1-\frac{(r-1)(\log n+1)}{n}}$$

Thus the expected number of trials to get $n/\log n$ clean candidates is:

$$\sum_{j=n-n/\log n}^{n} \frac{n}{\frac{n}{\log n}-(r-1)} \le \sum_{j=1}^{n} \frac{n}{\frac{n}{\log n}-(r-1)}$$

$$= \sum_{j=1}^{n} \frac{\log n+1}{1-\frac{(r-1)(\log n+1)}{n}}$$

$$= n \times \left(\frac{\log n + 1}{1 - \frac{(r-1)(\log n + 1)}{n}} \right)$$

$$= n\,(\log n + 1) \times \left(\frac{1}{1 - \frac{(r-1)(\log n + 1)}{n}} \right)$$

For a BST with K vertices we need to spend O(log K) time to verify a particular vertex's existence in the BST. So we multiply the above by log K. The expected number of trials to get n/log n clean candidates for 1 single source is

$$\leq n\,(\log n + 1) \times \log K \times \left(\frac{1}{1 - \frac{(r-1)(\log n + 1)}{n}} \right)$$

Thus the expected number of trials to get n/log n candidates for the n single source is $\leq n^2\,(\log n + 1) \times \log K \times \left(\frac{1}{1 - \frac{(r-1)(\log n + 1)}{n}} \right)$

Using Lemma 2 and setting $x = \frac{(r-1)(\log n + 1)}{n}$ we can write the following:

$$n^2 (\log n + 1) \times \log K \times \left(\frac{1}{1 - \frac{(r-1)(\log n + 1)}{n}} \right)$$

$$\leq n^2 (\log n + 1) \times \log K \times \left(1 + \frac{2(r-1)(\log n + 1)}{n} \right)$$

where $0 \leq \frac{(r-1)(\log n + 1)}{n} \leq \frac{1}{2}$

Thus the expected number of trials to get $n/\log n$ clean candidates for the n single source problems of 1 iteration of r is

$$\leq T_r = n^2 (\log n + 1) \times \log K \times \left(1 + \frac{2(r-1)(\log n + 1)}{n} \right)$$

Now we take the summation for $r = 1$ to K for K iterations.

$$\sum_{r=1}^{K} T_r = \sum_{r=1}^{K} n^2 (\log n + 1) \times \log K \times \left(1 + \frac{2(r-1)(\log n + 1)}{n} \right)$$

$$= \sum_{r=1}^{K} (n^2 \log n \log K + n^2 \log K) \left(1 + \frac{2(r-1)(\log n + 1)}{n} \right)$$

$$= \sum_{r=1}^{K} n^2 \log n \log K + \left(n^2 \log n \log k \times \frac{2(r-1)(\log n + 1)}{n} \right) +$$

$$n^2 \log K + \left(n^2 \log K \times \frac{2(r-1)(\log n + 1)}{n} \right)$$

$$= \sum_{r=1}^{K} n^2 \log n \log K + (n \log n \log K \times 2(r-1)(\log n + 1)) +$$

$$n^2 \log K + (n \log K \times 2(r-1)(\log n + 1))$$

Let $A^1 = n^2 \log n \log K$

$A^2 = n \log n \log K \times 2(r-1)(\log n + 1)$

$A^3 = n^2 \log K$

$A^4 = n \log K \times 2(r-1)(\log n + 1)$

Thus we can write the following:

$$= \sum_{r=1}^{K} A^1 + A^2 + A^3 + A^4$$

$$= \sum_{r=1}^{K} A^1 + \sum_{r=1}^{K} A^2 + \sum_{r=1}^{K} A^3 + \sum_{r=1}^{K} A^4$$

Let's do part A^1, A^2, A^3 and A^4 separately.

Part A^1:

$$\sum_{r=1}^{K} A^1 = \sum_{r=1}^{K} n^2 \log n \log K$$

$$= Kn^2 \log n \log K$$

Part A^2:

$$\sum_{r=1}^{K} A^2 = \sum_{r=1}^{K} n \log n \log K \times 2(r-1)(\log n + 1)$$

$$= n \log n \log K \times 2 \times \frac{K(K-1)}{2} \times (\log n + 1) \text{ (using Lemma 4)}$$

$$= n \log n \log K \times (K^2 - K) \times (\log n + 1)$$

$$= (K^2 n \log n \log K - Kn \log n \log K)(\log n + 1)$$

$$= K^2 n \log^2 n \log K - Kn \log^2 n \log K +$$

$$K^2 n \log n \log K - Kn \log n \log K$$

$$\leq K^2 n \log^2 n \log K + K^2 n \log n \log K$$

$$\leq K^2 n \log^2 n \log K$$

Part A^3:

$$\sum_{r=1}^{K} A^3 = \sum_{r=1}^{K} n^2 \log K$$

$$= Kn^2 \log K$$

Part A^4:

$$\sum_{r=1}^{K} A^4 = \sum_{r=1}^{K} n \log K \times 2(r-1)(\log n + 1)$$

$$= n \log K \times 2 \times \frac{K(K-1)}{2} \times (\log n + 1) \text{ (using Lemma 4)}$$

$$= n \log K (K^2 - K)(\log n + 1)$$

$$= (K^2 n \log K - Kn \log K)(\log n + 1)$$

$$= K^2 n \log n \log K - Kn \log n \log K + K^2 n \log K - Kn \log K$$

$$\leq K^2 n \log n \log K + K^2 n \log K$$

$$\leq K^2 n \log n \log K$$

Now we put part A^1, A^2, A^3 and A^4 together and get

$$\sum_{r=1}^{K} T_r = O(Kn^2 \log n \log K + K^2 n \log^2 n \log K + Kn^2 \log K + K^2 n \log n \log K)$$

$\leq O(Kn^2 \log n \log K + K^2 n \log^2 n \log K)$

$\leq O(Kn^2 \log n \log K)$ when $K \leq n/\log n$

Finally we can claim the expected number of trials to get $n/\log n$ clean candidates for the n single source problems of K iterations of r is $\leq O(Kn^2 \log n \log K)$ when $K \leq n/\log n$.

From these observations we can conclude the complexities before and after the critical point are balanced to be $O(n \log n \log K)$, resulting in the total expected time of $O(n \log n \log K)$. Thus the expected time for the n single source problems becomes $O(n^2 \log n \log K)$. As we consider K shortest distances from n single sources the complexity further becomes $O(Kn^2 \log n \log K)$ in total when $K \leq n/\log n$.

The time to build the initial distance vector before the critical point is $O(n \log K)$ as it takes $O(\log K)$ time to determine whether the vertex in the second layer we currently consider already exists in the BST at the $T[i][cand[k]$ location. And as it runs n times the complexity becomes $O(n \log K)$. Thus for the n single source K shortest paths problems the time to build the initial distance vector is $O(n^2 K \log K)$. But this complexity will be absorbed by the greater complexity in phase 1 and phase 2. Also the complexity of

initialization of $(n + 1)^{th}$ column of array T with positive infinite values would be absorbed by the overall complexity of the algorithm.

4.4 New K-MSP Algorithm

To incorporate Algorithm 11 into *K*-MSP we modify Takaoka and Bae's [2] *K*-MSP algorithm (which is based on sub-cubic DMM) and hence develop a new algorithm for *K*-MSP. This new algorithm is as follows:

Algorithm 12: New *K*-MSP Algorithm based on *K*-Tuple Approach

1: If the array becomes (a, a) return solution by Algorithm 13.
2: Otherwise, if $m > n$, rotate the array 90 degrees.
3: Thus we assume $m \leq n$.
4: Let A_{NW} be the solution for the NW part.
5: Let A_{NE} be the solution for the NE part.
6: Let A_{SW} be the solution for the SW part.
7: Let A_{SE} be the solution for the SE part.
8: Let A_{column} be the solution for the column-centered problem.
9: Let A_{row} be the solution for the row-centered problem.
10: Let the solution be the *K*-Tuple of *K* maxima selected from
11: $\{A_{NW} \cup A_{NE} \cup A_{SW} \cup A_{SE} \cup A_{column} \cup A_{row}\}$.

4.5 Description of New K-MSP Algorithm

In the following we describe Algorithm 12. At first we scrutinize the base case of the recursion. Then we describe the recursive part of the algorithm.

4.5.1 Base Case

In Algorithm 12, we deal with *K*-Tuple. To do so we find the new base condition of the recursion. When the recursion hits an (*a*, *a*) array we select *K* largest sums by an exhaustive method where *a* refers to the value that we have established in Lemma 6 (Part B). We slightly modify Algorithm 2 and represent this new exhaustive algorithm in Algorithm 13.

Algorithm 13: Modified Exhaustive Algorithm for MSP

```
 1: for i := 1 to m
 2:   for j := 1 to n
 3:     for k := 1 to i
 4:       for l:= 1 to j
 5:       currentmax = s[i][j] – s[i][l] – s[k][j] + s[k][l];
 6:       insert currentmax into the maximum subarray solution set;
 7:       end for;
 8:     end for;
 9:   end for;
10: end for;
```

The above algorithm exhaustively finds all possible subarrays for a given array. Using Lemma 6 (Part *B*), when we reach the bottom of the recursion with an (*a*, *a*) array in Algorithm 12 there are at most $K + a^3$ subarrays that can be selected. We use Algorithm 13 to find all possible subarrays for an (*a*, *a*) array. In this *K*-Tuple Approach as we need to return *K* items from each level of the recursion we need to find *K* best subarrays out of these $K + a^3$ subarrays. *K*-minima or *K*-maxima of $K + a^3$ elements can be found in *CK* comparisons

where C is a constant. To do so we incorporate the linear time selection algorithm in [14] for selecting the K^{th} element from n elements. Once the K^{th} element is found we can select K-minima or K-maxima by filtering n elements by comparing them with this element. Thus in CK comparisons we can find K-minima or K-maxima of $K + a^3$ elements where the value of C is determined by the selection algorithm we mentioned above.

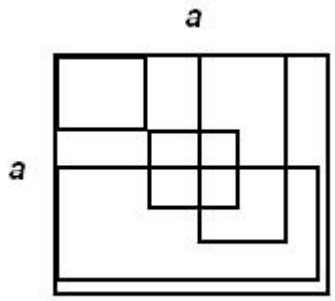


Figure 19: (a, a) array at the bottom of the recursion

4.5.2 Recursive Case

Now we focus into the recursive part of Algorithm 12. We define 6 conditional solutions for MSP. Let us divide the array by 4 equal parts by central horizontal and central vertical line as in the following figure.

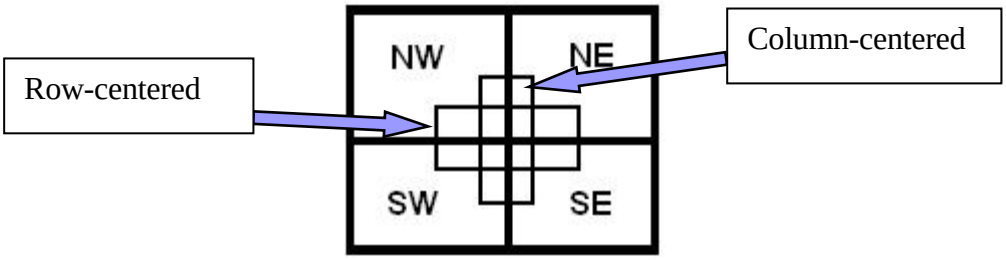


Figure 20: Four-way recursion and 6 solutions

A_{NW} = Maximum subarray to be found in the Upper-Left part

A_{NE} = Maximum subarray to be found in the Upper-Right part

A_{SW} = Maximum subarray to be found in the Lower-Left part

A_{SE} = Maximum subarray to be found in the Lower-Right part

A_{column} = Maximum subarray to be stretched over the central vertical line

A_{row} = Maximum subarray to be stretched over the central horizontal line

A_{column} can be solved using Equation (1) as before. Also A_{row} can be solved in the same way as row-centered problem is a symmetrical problem to column-centered problem. Thus we convert MSP into DMM. Finally when we apply Equation (2) we realize that each component of Equation (2) is now a set of *K*-Tuple. We rewrite Equation (2) in the following line.

$$S_2 S_2^* - S_1 S_1^*$$

Since each component of Equation (2) is a set of *K*-Tuple the matrix subtraction is computed by $a - b$ operation component-wise. Now we define the subtraction of two sets of *K*-Tuples where *K* MAX $\{a\}$ selects *K* largest elements in a set *a*.

$$a - b = K \text{ MAX}\{a_i - b_j \mid a_i \in a, b_j \in b, 1 \leq i, j \leq K\}$$

According to Frederickson and Johnson [15], selection of *K* largest elements in Cartesian sum $X + Y$ is solved in O(*K*) time where $|X| = |Y| = K$. In the following we describe an O(*K* log *K*) method for $X + Y$ problem with an example. The same method can be used for $a - b$ subtraction component wise.

For instance, we are given two sets *X* and *Y* as follows.

$X = \{2, 8, 7\}$ $Y = \{3, 1, 6\}$

We need to first sort *K* items of both sets in ascending order. Using heap sort we can sort *K* items in O(*K* log *K*) time. After sorting we have,

$X = \{2, 7, 8\}$ $Y = \{1, 3, 6\}$

Then we select the minimum items from both sets. These are naturally first 2 items of both sets. We add these 2 items and put the resulting value with their

corresponding index values into Binary Heap. So 2 + 1 = 3 is inserted in the heap with index values (1, 1). Then we find the minimum value from the heap and extract it out of the heap. We return the minimum value that we extracted from the heap but we save the index values for our next operation. So keep track of (1, 1) for which 3 came for. Next, we add 1 to each index value separately to insert new items into the heap. New index values are (1 + 1, 1) = (2, 1) and (1, 1 + 1) = (1, 2). We add items in both index values and insert into the heap. For index value (2, 1) we get 7 + 1 = 8 and we insert this into the heap.

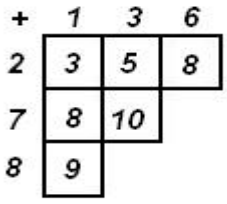


Figure 21: *X* + *Y* problem solving in O(*K* log *K*) time

For index value (1, 2) we get 2 + 3 = 5 and we insert this into the heap as well. Then we find the minimum value from the heap and return it. This process keeps going until we find *K* minima.

Finding *K* items from the heap would cost us again another O(*K* log *K*) time. The complexity of *X* + *Y* problem becomes O(*K* log *K* + *K* log *K*). Finally

since both components are of the same order one will be absorbed with the other. So the total complexity for $X + Y$ problem becomes $O(K \log K)$.

4.6 Analysis of New K-MSP Algorithm

We assume m and n are each a power of 2, and $m \leq n$. By chopping the array into squares we can consider the case where $m = n$. Let $T(n)$ be the time to analyze an array of size (n, n). Algorithm 12 splits the array vertically and horizontally. We multiply $(n, n/2)$ and $(n/2, n)$ matrices by 4 multiplications of size $(n/2, n/2)$ and analyze the number of comparisons. Let $M(n)$ be the time for multiplying two $(n/2, n/2)$ matrices which is equal to $O(Kn^2 \log n \log K)$ as $O(Kn^2 \log n \log K)$ is the expected time for the n single source K shortest paths problems when $K \leq n/\log n$. Thus we can establish the following lemma, recurrence and theorem.

4.6.1 Lemma 9

If $M(n) = O(Kn^2 \log n \log K)$ then $M(n)$ satisfies the following condition:

$M(n) >= (4 + 4 / \log n)M(n/2)$.

Proof:

$M(n) = Kn^2 \log n \log K$

$$= K \log K \times 4 \left(\frac{n}{2}\right)^2 \log \left(2.\frac{n}{2}\right)$$

$$= K \log K \times 4 \left(\frac{n}{2}\right)^2 \left(1 + \log \frac{n}{2}\right)$$

$$= K \log K \times 4 \left(\frac{n}{2}\right)^2 \log \frac{n}{2} \left(1 + \frac{1}{\log \frac{n}{2}}\right)$$

$$= \left(4 + \frac{4}{\log \frac{n}{2}}\right) \mathrm{M}\left(\frac{n}{2}\right) > \left(4 + \frac{4}{\log n}\right) \mathrm{M}\left(\frac{n}{2}\right)$$

Thus $M(n)$ satisfies the condition.

4.6.2 Recurrence 2

Let T_1, T_2, ….., T_N be the times to compute DMMs of different sizes in the Algorithm 12. Then by ignoring some overhead time between DMMs, the expected value $E[T]$ of the total time T becomes

$E[T] = E[T_1 + T_2 + \ldots\ldots + T_N]$

$= E[T_1] + E[T_2] + \ldots\ldots + E[T_N]$

$= E[E[T_1 \mid T_2, T_3, \ldots, T_N]] + E[E[T_2 \mid T_1, T_3, \ldots, T_N]] + \ldots + E[E[T_N \mid T_1, T_2 \ldots, T_{N-1}]]$

From the theorem of total expectation, we have $E[E[X|Y]] = E[X]$ where $X|Y$ is the conditional random variable of X conditioned by Y. So we can write

$$E[T] = E_{T_1}[T_1] + E_{T_2}[T_2] + \ldots + E_{T_N}[T_N]$$

In the above, expectation operators go over the sample space of $T_1, T_2 \ldots T_N$. Thus we can establish the following recurrence where total expected time is expressed as the sum of the expected times of algorithmic components.

$$T(a) = CK,\ T(n) = 4T(n/2) + 16C\,M(n)$$

$$M(n) = O(Kn^2 \log n \log K)$$

4.6.3 Theorem 2

Suppose $M(n) = O(Kn^2 \log n \log K)$ and $M(n)$ satisfies the condition $M(n) \geq (4 + 4 / \log n)\, M(n/2)$. Then the above $T(n)$ satisfies the following:

$$T(n) \leq 16C\,(1 + \log n)\, M(n) \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Proof:

*Basis step***:** By setting $n = a$ into Equation (3) we get

L.H.S. = $T(a) = CK$

R.H.S. = $16C\ (1 + \log a)\ M(a)$

We need to show L.H.S. $\leq$ R.H.S.

R.H.S. = $16C\ (1 + \log a)\ M(a)$

$= 16C\ (1 + \log a)\ K(a)^2 \log a \log K$

$\geq 16CK \log K$

$\geq CK$

So L.H.S. $\leq$ R.H.S. *(proved)*

Thus Theorem 2 holds for $T(a)$.

Inductive step:

$T(n) = 4T(n/2) + 16C\ M(n)$

$= 4 \times 16C\ (1 + \log \frac{n}{2})\ \mathrm{M}(\frac{n}{2}) + 16C\ M(n)$

$$\leq 4 \times 16C\,(1 + \log\frac{n}{2})\left(\frac{M(n)}{4+\frac{4}{\log n}}\right) + 16C\,M(n)$$

$$= 64C\,(1 + \log n - \log 2)\left(\frac{M(n)}{4\left(1+\frac{1}{\log n}\right)}\right) + 16C\,M(n)$$

$$= 16C\,(1 + \log n - 1)\left(\frac{M(n)}{\frac{\log n+1}{\log n}}\right) + 16C\,M(n)$$

$$= 16C\log n\left(\frac{M(n)}{\frac{\log n+1}{\log n}}\right) + 16C\,M(n)$$

$$= 16C\log^2 n\left(\frac{M(n)}{\log n+1}\right) + 16C\,M(n)$$

$$= 16C\,(\log^2 n + \log n - \log n)\left(\frac{M(n)}{\log n+1}\right) + 16C\,M(n)$$

$$= 16C\,(\log n\,(\log n + 1) - \log n)\left(\frac{M(n)}{\log n+1}\right) + 16C\,M(n)$$

$$= 16C\left(\frac{\log n(\log n+1)-\log n}{\log n+1}\right)M(n) + 16C\,M(n)$$

$$= 16C\left(\log n-\frac{\log n}{\log n+1}\right)M(n) + 16C\,M(n)$$

$$\leq 16C \log n\, M(n) + 16C\, M(n)$$

$$= 16C\, M(n)\,(1 + \log n)$$

$$= 16C\,(1 + \log n)\, M(n) \quad \textit{(proved)}$$

We perform K-DMM by Algorithm 12 and solve the MSP by Algorithm 13. The complexity of K-DMM is $O(Kn^2 \log n \log K)$ when $K \leq n/\log n$ and K-Tuple matrix subtraction is $O(K \log K)$. The complexity of K-Tuple subtraction will be absorbed by the complexity of K-DMM. As we solve the K-MSP recursively there will be an extra $\log n$ factor. Thus the total complexity for K-MSP becomes $O(Kn^2 \log^2 n \log K)$ when $K \leq n/\log n$.

Chapter 5

Tournament Approach

In this chapter we develop a Tournament algorithm for *K*-MSP based on Tournament Approach and achieve better time complexity over *K*-Tuple Approach.

5.1 Tournament Algorithm

A classical selection problem is that of selecting *K* smallest or *K* largest elements out of *n* elements. This problem can be easily explained by a typical knock-out tournament. Suppose in a knock-out tournament there are 8 players going to participate. The basic idea is to have a knock-out minimal round tournament to trace the winner. We construct the following tournament tree to

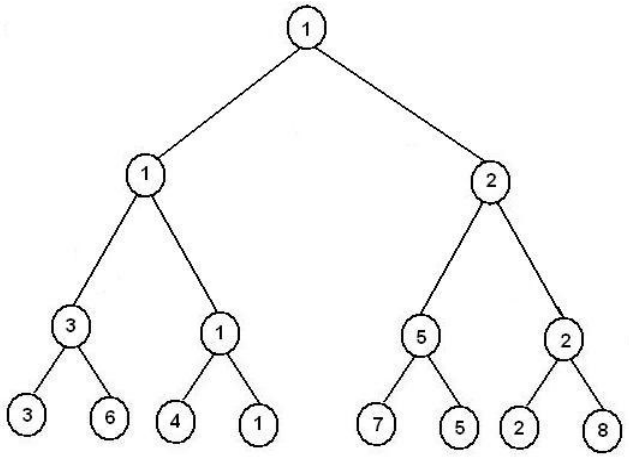


Figure 22: Binary tournament tree

find the winner. For simplicity we identify the player by the player's key value and smaller the key value stronger the player is.

We organize the matches between adjacent pairs and we move the winners to the next level of the tournament until the champion of the tournament is found. We can observe player 1 beats player 4, 3, 2 and moves up to the root of the tree as the winner. If we are concerned about the second best player of the tournament we would select player 2 by conventional approach as player 2 was beaten by player 1 (winner) in the last round. This approach is currently in practice and can be observed in today's knock-out tournament of any game. But the way we select the second best player is really an unjustified manner and this was first mentioned by Dodgson [16] who is better known as Lewis Carroll. The fact that Dodgson pointed out was all the players that were beaten directly by the tournament winner should be considered to compete for the second position. According to this, in the above example player 2, 3 and 4 all have the equal right to compete for the second position. Because player 3 and 4 were directly beaten by the superior player 1 only and they were not beaten by player 2. So there is no justification of choosing player 2 as the second best. So another tournament is required to choose the second best among these potential players who were directly beaten by the winner of the tournament.

Dodgson designed a Tournament Approach to determine the true second best, third best and so on until the K^{th} best and it was further mentioned and explained by Don Knuth in [17]. Tournament algorithm constructs the binary tree as above. The second player must be the one from all the direct losers to the tournament winner. These potential second best players can be found by walking in the binary tree in O(log n) time. After finding all the potential second best players another tournament is to be arranged among these players to find the true second best player. The third best potential players are those who were a direct loser to the second best player in any of the previous 2 tournaments. Thus the total complexity to find K best players becomes O(n + K log n).

5.2 Tournament Approach for K-MSP

K-Tuple Approach that we have described in the previous chapter is a heavy-handed approach as we maintain K elements all the way to the bottom of the recursion. We maintain a tuple set S of K elements after every DMM we perform. And this process of calculating and maintaining the tuple set S of K elements is repeated for each maximum subarray we calculate. That is, we spend effort for second, third, fourth, ……, K^{th} maximum subarray as much as we spend for the first maximum subarray. In the end, the total complexity of

K-MSP becomes a multiplication of the effort required to calculate 1 maximum subarray by the number of maximum subarrays to be searched for.

In Tournament Approach, instead of spending same effort for all *K* maximum subarrays we spend some extra effort during the calculation of the first maximum subarray and in return we spend less effort during the calculation of all subsequent subarrays. The basic idea came from the realization of the fact that second maximum subarray will be found based on the first maximum subarray. Then third maximum subarray will be found based on the first and second maximum subarrays. After finding the first maximum subarray when we search for the second maximum subarray, if we can locate exactly from which location of the whole array the first maximum subarray came from we can spend less effort to find the second maximum subarray by only considering all the candidates that were beaten by the first maximum subarray solution.

We can visualize the recursion process of Algorithm 12 in the following figure. At every recursion level there are 6 solutions. Out of these 6 solutions the best one gets carried through to the upper level of recursion. The 4 solutions that are depicted as circles in the following figure are A_{NW}, A_{NE}, A_{SW} and A_{SE}. The solutions that are depicted as rectangles are column-centered and row-centered solutions. The final solution that is carried through to the top of

the tree must come from some level in the recursion and it must be either a row-centered solution or a column-centered solution or a single element at the bottom.

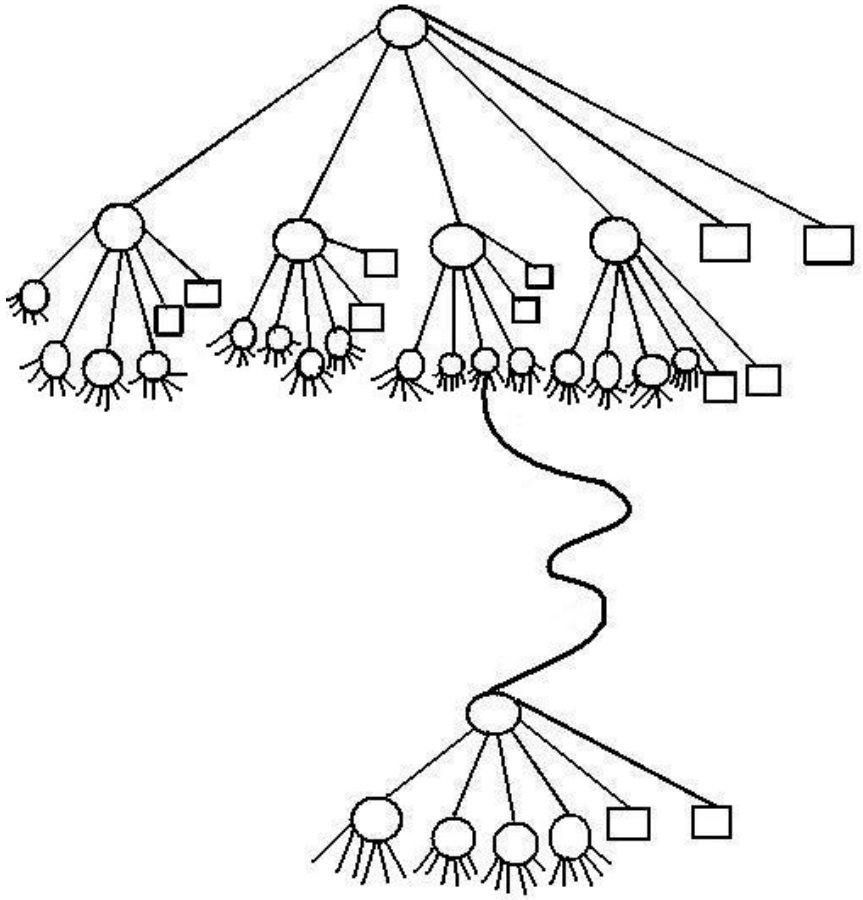


Figure 23: Visualization of the intermediate recursion process

Suppose in the above figure, we have detected the recursion level from which the final solution returned. Also we've identified that it was a column-centered solution that was carried up in the recursion. In the following figure we

enlarge this column-centered solution by focusing all possible strips inside the column-centered solution.

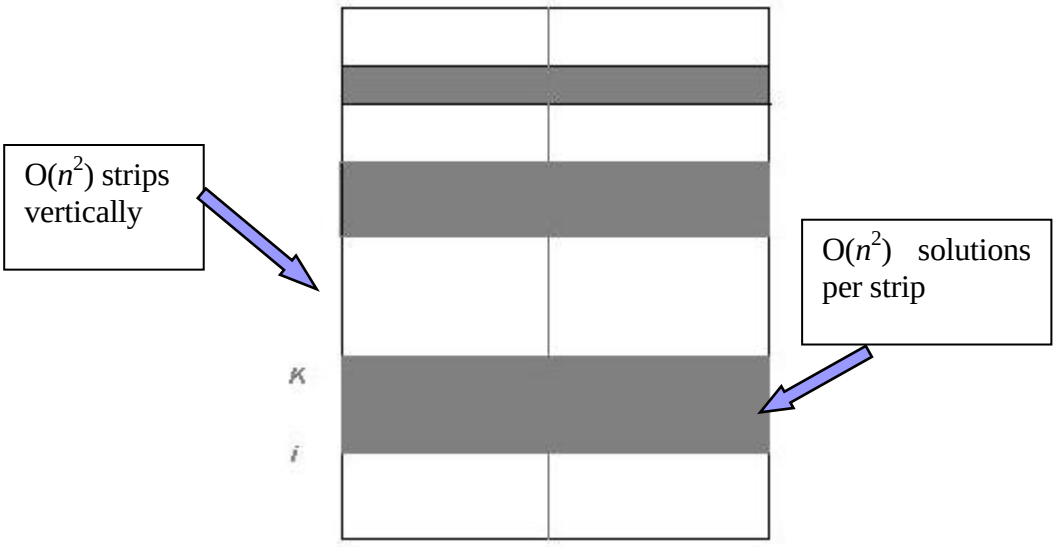


Figure 24: Strip Visualization

For simplicity we have drawn only 3 strips in the above picture for the column-centered solution. For all values of index *k* and *i* there are O(n^2) strips from which the column-centered solution might have come up. Also within a strip there are O(n^2) possible subarrays from which the column-centered solution might have come up. We discuss this in details later in this chapter.

We develop the following algorithm which is a modified version of Algorithm 12 for finding the first maximum subarray. We identify the solution by the value of the sum in the maximum subarray, the indices to indicate where this

array is and the portion where the solution column-centered or row-centered was calculated. The position of the solution is represented by 4 indices *i*, *j*, *k*, *l* and scope of the solution is represented by 4 indices as well which are *m*1, *n*1, *m*2 and *n*2.

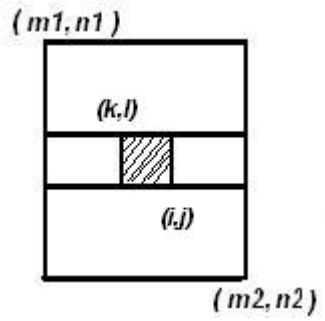


Figure 25: Scope and position of a maximum subarray solution

Algorithm 14: New *K*-MSP Algorithm based on Tournament Approach

1: If the array becomes one element, return its value.
2: Otherwise, if $m > n$, rotate the array 90 degrees.
3: Thus we assume $m \leq n$.
4: Let A_{NW} be the solution for the NW part.
5: Let A_{NE} be the solution for the NE part.
6: Let A_{SW} be the solution for the SW part.
7: Let A_{SE} be the solution for the SE part.
8: Let A_{column} be the solution for the column-centered problem.
9: Let A_{row} be the solution for the row-centered problem.
10: Let *S* be the maximum of those six.
11: Return *S* along with scope, position and recursion depth of *S*.

Algorithm 10 (Fast DMM) is to be incorporated in the above algorithm for performing DMM. During the process of finding the first maximum subarray

by Algorithm 14 we do some additional works as well. While we solve either column-centered problem or row-centered we perform DMM and we compare all the best values found among all the n^2 possible strips.

After we locate the scope of the first solution the size of the scope is denoted by n. In one single strip horizontally there are $O(n^2)$ possible subarrays and precisely $\frac{n^2}{4}$ subarrays. All $O(n^2)$ subarrays compete with each other to become the second winner of the strip. This can be done in $O(n^2)$ time. Also there are vertically $O(n^2)$ possible strips for a given column-centered problem. So there are maximum $O(n^2)$ possible winners for $O(n^2)$ possible strips. In $O(n^2)$ time we can find the overall second winner strip among all $O(n^2)$ strips.

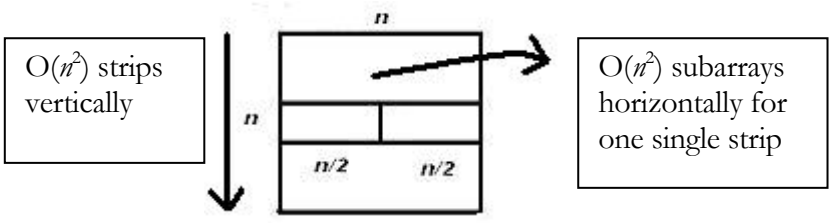


Figure 26: (n, n) array yielding $O(n^2)$ strips and $O(n^2)$ subarrays per strip

Once we calculate the winner of a particular strip we keep track of the position of the winner solution within the strip by recording the co-ordinates information of the winner for later use. The complexity to keep track of co-

ordinates of the winner of a particular strip is absorbed with in the complexity of finding the winner of that strip. In this way, the winner solution from a winner strip is returned as column-centered problem solution.

When we complete finding the first maximum subarray we return the solution along with solution's scope, position and recursion depth. We now consider the situation of finding the second maximum subarray. Using the recursion depth information of the first maximum subarray we straightaway locate the recursion level. Then using scope and position information of the last solution we can exactly find where the last solution came from and thus we can detect which 1 of the 6 solutions was responsible for the last final solution. All 6 solutions competed with each other and eventually the best one was carried up the recursion. Let us assume column-centered solution was the winner. The basic idea that we apply here using Tournament Approach is that all the 5 solutions that were beaten by the winner column-centered solution from that recursion level deserve to be the second best solution from that recursion level. Also the second best solution for the column-centered problem participates in this race. By using the co-ordinates information of the winners of all strips that we accumulated during the calculation of first maximum subarray we can find the second best column-centered solution. We compare the new column-

centered solution with other 5 solutions and pick up the best solution and feed up through the recursion for the second maximum subarray calculation.

Finding the second best solution for a column-centered or row-centered problem can be done in $O(n^2)$ time. As we do this $K - 1$ times for $K - 1$ subsequent maximum subarrays the complexity for this becomes $O(Kn^2)$.

Thus the total expected time of K-MSP by Tournament Approach becomes $O(n^2 \log^2 n + Kn^2)$.

Chapter 6

Evaluation

In this chapter we present time measurements of Conventional (exhaustive) Approach, *K*-Tuple Approach and Tournament Approach. All the algorithms were implemented in programming language C. In the following sections we focus on time measurements of a number of algorithms and we intentionally omit implementation details of these algorithms. First we compare Conventional Approach with *K*-Tuple Approach by investigating the result of the experiments. Then we compare *K*-Tuple Approach with Tournament Approach in the same way.

6.1 Conventional Approach vs. K-Tuple Approach

In the beginning, Conventional DMM (exhaustive method) was implemented. Then we extended this to Conventional *K*-DMM. Also we implemented Algorithm 6 for MSP where we incorporated Conventional DMM to solve column-centered and row-centered problems. After this we implemented Algorithm 12 for *K*-MSP where we incorporated exhaustive *K*-DMM algorithm for column-centered and row-centered problems. The time measurement was taken and recorded for further comparison with efficient methods. In all of the above cases we fixed $K = 10$.

Then Fast version of DMM was implemented which is basically incorporation of MT [3] APSP Problem algorithm into DMM. So MT algorithm was needed to investigate first. Then again MT is based on Dantzig's [10] and Spira's [11] APSP Problem algorithms. So Dantzig's and Spira's APSP Problem algorithms were implemented before we implemented MT. Large amount of random data were created to measure the expected time. After this, MT was efficiently implemented. Also Dijkstra's [25] and Floyd's [26] APSP Problem algorithms were implemented and compared with Dantzig, Spira and MT just for the sake of interest. Then we extended MT to *K*-DMM and implemented Algorithm 11 and the expected time was measured. Binary heap was used for the priority queue. In the end, this Fast *K*-DMM was incorporated into *K*-MSP algorithm which is Algorithm 12. Again the value of *K* was fixed to 10 in all of the above cases.

6.1.1 Dijkstra, Floyd, Dantzig, Spira and MT APSP Problem Algorithms

Size (n)	Dijkstra (sec)	Floyd (sec)	Dantzig (sec)	Spira (sec)	MT(sec)
100	0.03	0.02	0.03	0.03	0.02
200	0.26	0.16	0.17	0.15	0.12
300	0.93	0.55	0.52	0.39	0.28
400	2.07	1.24	1.15	0.66	0.50
500	4.66	2.92	2.27	1.03	0.83
600	7.46	4.45	3.63	1.94	1.33
700	12.87	7.99	5.63	2.79	1.92

800	18.18	11.18	8.01	3.70	2.57
900	26.03	15.93	11.15	4.05	3.07
1000	36.85	23.18	15.10	5.06	3.97

Table 1: APSP Problem time measurement

In table 1 we observe MT outplays all others when input size is 200. As input size gets bigger we also observe MT becomes more efficient than all other algorithms compared above. The corresponding graph is shown in Figure 27.

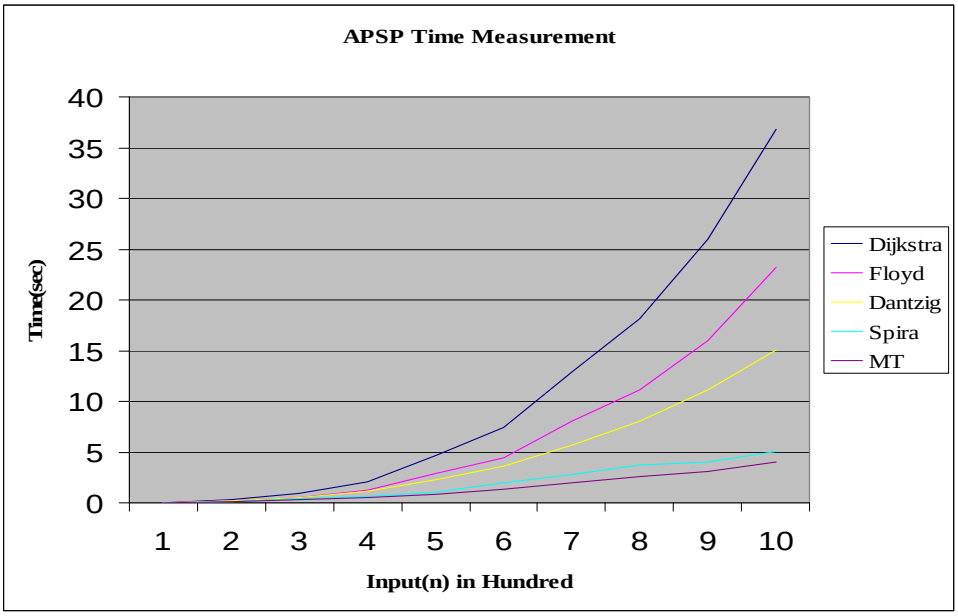


Figure 27: APSP Problem time measurement

6.1.2 Conventional DMM vs. Fast DMM

In Table 2 we observe the Fast DMM outplays the Conventional DMM when input size is 300. A significant difference between these two can be observed when input size is 400. As input size gets larger we also observe that Fast DMM completely outplays Conventional DMM. The corresponding graph is shown in Figure 28.

Size(n)	Conventional DMM (sec)	Fast DMM (sec)
100	0.02	0.03
200	0.15	0.16
300	0.55	0.51
400	1.45	1.12
500	3.22	2.25
600	5.02	3.62
700	8.89	5.83
800	13.26	8.23
900	19.83	11.51
1000	27.38	15.28

Table 2: DMM time measurement

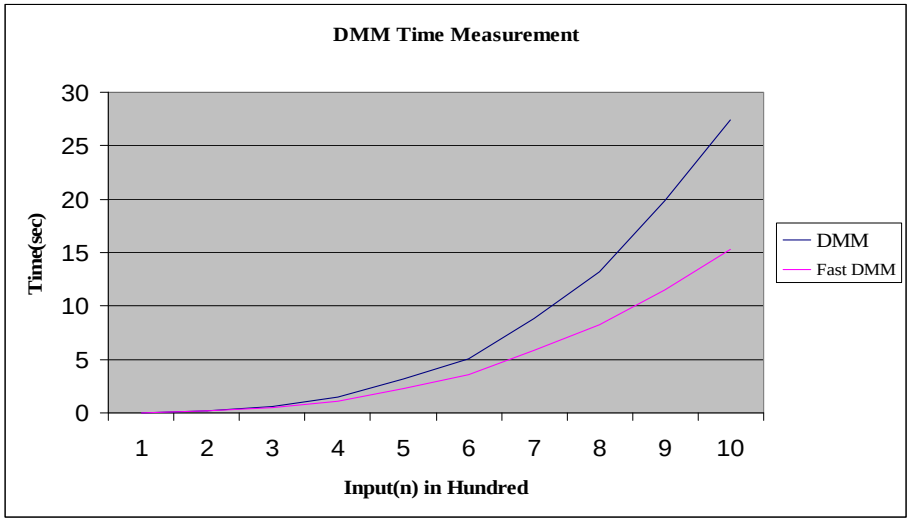


Figure 28: DMM time measurement

6.1.3 MSP with Conventional DMM vs. MSP with Fast DMM

Size(*n*)	MSP(with Conventional DMM) (sec)	MSP(with Fast DMM) (sec)
100	0.14	0.76
200	1.06	1.40
300	3.19	3.35
400	8.48	8.62
500	19.35	19.45
600	29.53	27.42
700	45.08	39.14
800	76.50	62.14
900	114.92	97.20
1000	157.46	130.2

Table 3: MSP time measurement

In the above table, we observe MSP with Fast DMM outplays MSP with Conventional DMM when input size is 600. When input size is 700 we observe a significant difference between two versions of MSP. One interesting observation is that due to recursion in the main MSP algorithm we've one extra log *n* factor which delays MSP incorporating Fast DMM to take over MSP incorporating Conventional DMM. We've seen previously Fast DMM outplays Conventional DMM significantly when input size was 400. But after incorporating this Fast DMM into MSP algorithm we inspect a significant difference between two versions of MSP when input size was 700. It is due to the overhead of recursive calls to Fast DMM from the main MSP algorithm. The corresponding graph is shown in Figure 29.

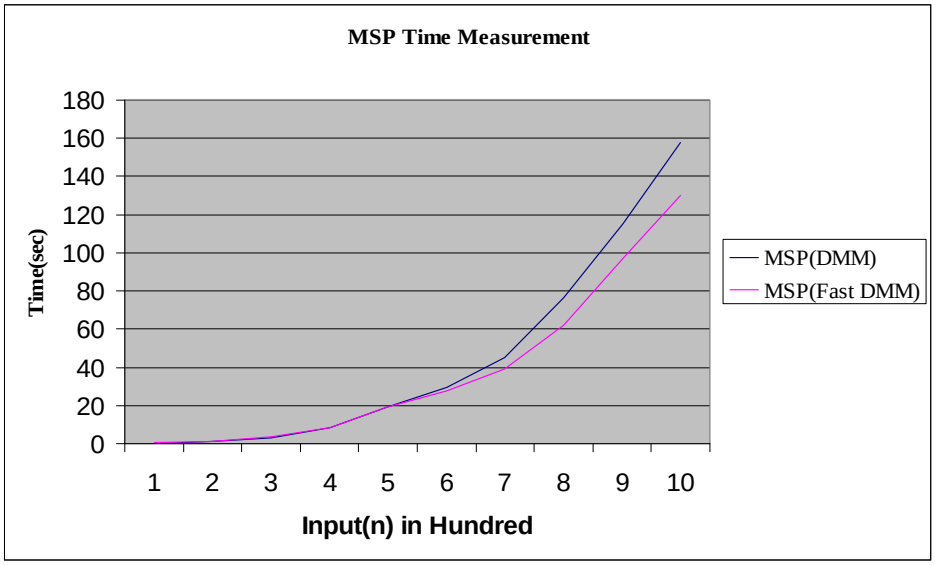


Figure 29: MSP time measurement

6.1.4 Conventional K-DMM vs. Fast K-DMM

In the above table we observe the Fast *K*-DMM outplays the Conventional *K*-DMM when input size is 200. A significant difference between these two can be observed when input size is 500. As input size gets larger we also observe that Fast *K*-DMM completely outplays Conventional *K*-DMM. The corresponding graph is shown in Figure 30.

Size(*n*)	Conventional *K*-DMM (sec)	Fast *K*-DMM (sec)
100	0.19	0.22
200	1.50	1.26
300	5.18	3.80
400	12.42	8.20
500	24.80	16.30
600	41.80	26.90
700	68.34	40.14
800	103.10	54.70
900	147.96	72.39
1000	209.59	93.75

Table 4: *K*-DMM time measurement

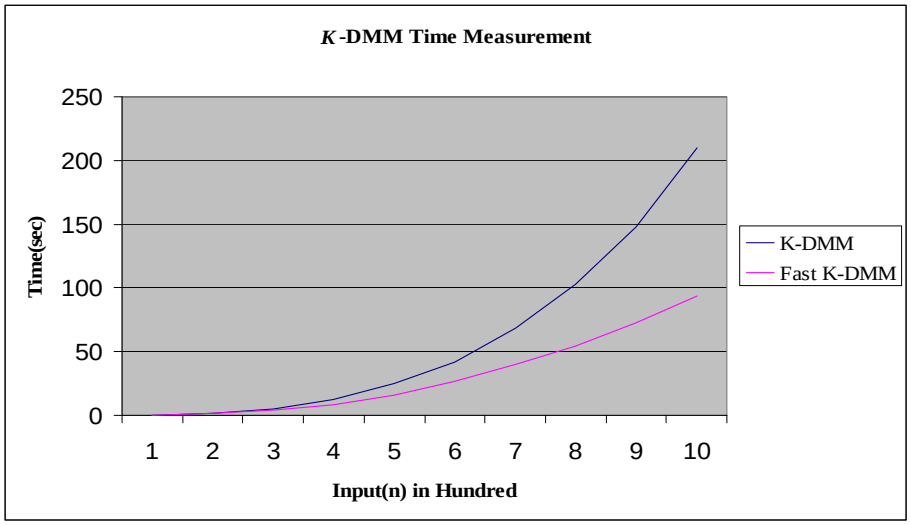


Figure 30: *K*-DMM time measurement

6.1.5 K-MSP with Conventional K-DMM vs. K-MSP with Fast K-DMM

In Table 5, we observe *K*-MSP with Fast *K*-DMM outplays *K*-MSP with Conventional *K*-DMM when input size is 700. But this difference remains subtle until input size reaches 800. When input size is 800 we observe a significant difference between the two versions of MSP.

Size(*n*)	*K*-MSP(with Conventional *K*-DMM) (sec)	*K*-MSP(with Fast *K*-DMM) (sec)
100	1.35	2.76
200	5.06	10.40
300	12.30	19.35
400	27.5	36.62
500	52.35	57.90

600	71.53	75.55
700	110.08	101.14
800	159.5	140.14
900	225.92	190.2
1000	305.46	245.2

Table 5: *K*-MSP time measurement (comparison between Conventional and *K*-Tuple Approach)

Interestingly, we observe that due to recursion in the main MSP algorithm we incur one extra log *n* factor which delays *K*-MSP incorporating Fast *K*-DMM to take over *K*-MSP incorporating Conventional *K*-DMM. We've seen previously Fast *K*-DMM outplays Conventional *K*-DMM significantly when input size was 500. But after incorporating this Fast *K*-DMM into *K*-MSP algorithm we see a significant difference between two versions of *K*-MSP when input size was 800. It is due to the overhead of recursive calls to Fast *K*-DMM from the main *K*-MSP algorithm. The corresponding graph is depicted in Figure 31.

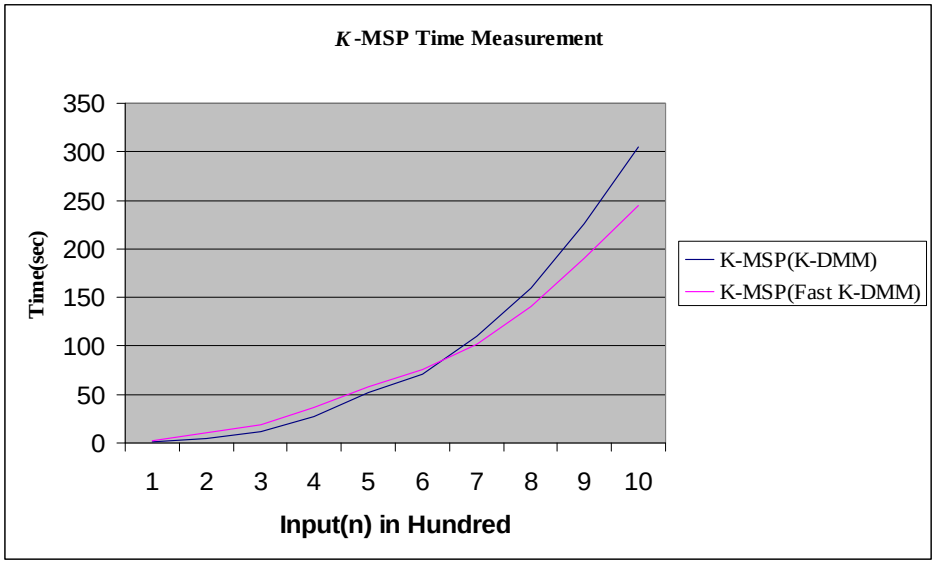


Figure 31: *K*-MSP time measurement (comparison between Conventional and *K*-Tuple Approach)

6.2 K-Tuple Approach vs. Tournament Approach

Finally, Tournament Approach was implemented by incorporating Fast DMM into the modified *K*-MSP algorithm that was discussed in Chapter 5. Note that the main purpose of Tournament Approach is to avoid *K*-Tuple Approach. So in this approach we don't need to incorporate *K*-DMM anymore. We incorporate Fast DMM into *K*-MSP as we discussed in Chapter 5.

In the following we present the time measurement of Tournament Approach and compare it with *K*-Tuple Approach.

Size(n)	K-Tuple Approach for K-MSP(sec)	Tournament Approach for K-MSP (sec)
100	2.76	1.76
200	10.40	8.40
300	19.35	16.35
400	36.62	29.62
500	57.90	49.90
600	75.55	66.42
700	101.14	91.00
800	140.14	128.14
900	190.2	175.2
1000	245.2	227.2

Table 6: *K*-MSP time measurement (comparison between *K*-Tuple and Tournament Approach)

In the above table, we observe the difference between the performances of *K*-Tuple Approach and Tournament Approach for *K*-MSP. Tournament Approach right from the beginning outplays *K*-Tuple Approach. When input size is 1000 we observe a significant difference between two versions of *K*-MSP. The corresponding graph is depicted in Figure 32.

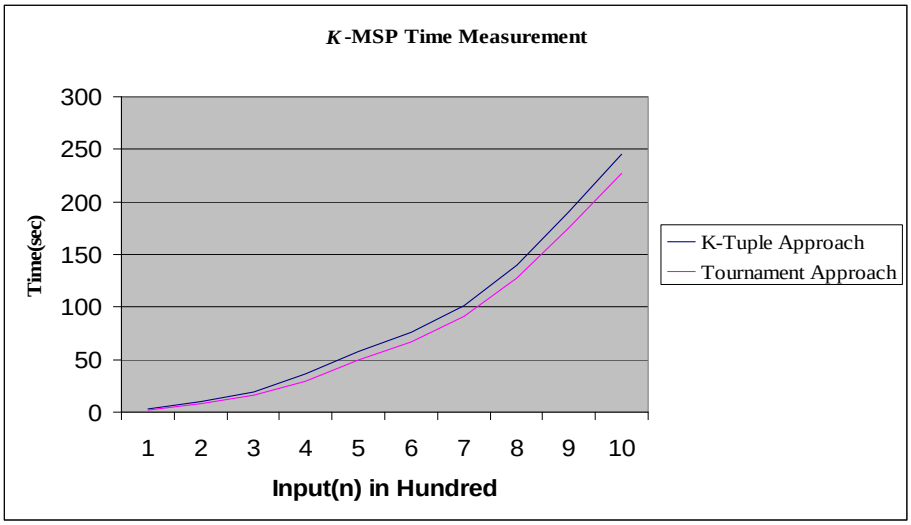


Figure 32: *K*-MSP time measurement (comparison between *K*-Tuple and Tournament Approach)

Chapter 7

Conclusion

In this research we have investigated MSP by doing the average case analysis. The basic aim of this research was to analyze existing algorithms for both MSP and *K*-MSP and develop efficient algorithms for *K*-MSP based on existing framework and algorithms. And finally the goal was to compare existing algorithms with the new algorithms based on time measurement.

We extended Fast DMM to Fast *K*-DMM and incorporated into MSP. To be able to incorporate Fast *K*-DMM into MSP we modified existing MSP and extended to *K*-MSP. With the help of advanced data structure we achieved $O(Kn^2 \log n \log K)$ time complexity for *K*-DMM when $K \leq n/\log n$. When we incorporated this *K*-DMM into the *K*-MSP an extra $\log n$ effort was required due to the recursion. So the overall complexity for *K*-MSP became $O(Kn^2 \log^2 n \log K)$ when $K \leq n/\log n$ based on *K*-Tuple Approach.

After this we realized *K*-Tuple Approach was a heavy-handed approach since every single element becomes a set of *K* items. We further reviewed this approach and came up with Tournament Approach. By Tournament Approach we solved the *K*-MSP more efficiently. The basic intention was to reuse the

information that we gain while we solve the first maximum subarray. By doing so, all the subsequent maximum subarrays can be detected in an efficient manner. And we showed how the complexity of this approach further comes down to $O(n^2 \log^2 n + Kn^2)$.

7.1 Future Work

The main framework of MSP remains unchanged in this research. Because of the recursion an extra log *n* effort is required when we use this framework. If we could change the main framework we may be able to remove this extra log *n* effort and reduce the overall complexity of both MSP and *K*-MSP. Another interesting research problem would be doing average case analysis of *K*-MSP for the disjoint case. Also in the Tournament Approach, with the help of advanced data structure finding the next best sum strip-wise could be done more efficiently and it would be really a challenging problem.

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